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Lower Bounds for $K_2^{\mathrm{top}}(\hat{\mathbb{Z}}_p\pi)$ and $K_2(\mathbb{Z}\pi)$

ROBERT OLIVER

Matematisk Institut, Ny Munkegade, DK-8000 Aarhus C, Denmark

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As seen in [11] and [4], the groups $K_2(\hat{\mathbb{Z}}_p\pi)$ (p a prime and π a finite group) play a role both in understanding $SK_1(\mathbb{Z}\pi)$ and in getting lower estimates for the size of $K_2(\mathbb{Z}\pi)$ or $Wh_2(\mathbb{Z}\pi)$. In particular, it was shown in [11] that the leading problem for computing $SK_1(\mathbb{Z}\pi)$ for arbitrary finite π is to understand $K_2(\hat{\mathbb{Z}}_p\pi)$ for all primes p and p-groups π. In this paper, we detect elements in $K_2(\hat{\mathbb{Z}}_p\pi)$ by describing the groups

$$K_2^*(\hat{\mathbb{Z}}_p\pi) = \varprojlim \mathrm{Coker}[K_2(\hat{\mathbb{Z}}_p\tilde{\pi}) \to K_2(\hat{\mathbb{Z}}_p\pi)]$$

and

$$Wh_2^*(\hat{\mathbb{Z}}_p\pi) = \varprojlim \mathrm{Coker}[Wh_2(\hat{\mathbb{Z}}_p\tilde{\pi}) \to Wh_2(\hat{\mathbb{Z}}_p\pi)],$$

where the limits are taken over all surjections of p-groups $\tilde{\pi} \twoheadrightarrow \pi$. In other words, we look at $K_2(\hat{\mathbb{Z}}_p\pi)$ and $Wh_2(\hat{\mathbb{Z}}_p\pi)$ after dividing out by all elements coming from arbitrarily large p-groups mapping onto π.

More generally, we let A be any unramified p-ring—the ring of integers in any finite unramified extension K of $\hat{\mathbb{Q}}_p$—and study the groups $K_2^*(A\pi)$ and $Wh_2^*(A\pi)$ defined analogously for any p-group π. In order to state the main result, we define

$$\mathscr{U}(A\pi) = H_1(\pi; A\pi)/\langle g \otimes \lambda g^n : g \in \pi, \lambda \in A, n \in \mathbb{Z} \rangle,$$

where π acts on $A\pi$ via conjugation. Then

$$\mathscr{U}(A\pi) \cong \sum_{i=1}^{k} (Z_\pi(g_i)/S_\pi(g_i))^{ab} \otimes A(g_i),$$

where $g_1, \ldots, g_k$ are conjugacy class representatives for π, $Z_\pi(g_i)$ is the centralizer, and

$$S_\pi(g_i) = \langle h \in \pi : h^n = g_i, \text{ some } n \rangle.$$

There is a map

$$v: \mathscr{U}(A\pi) \to H_2(\pi)$$

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defined by setting $v(h \otimes \lambda g) = \mathrm{Tr}(\lambda) \cdot (h \wedge g)$ (where $h \wedge g \in H_2(\pi)$ is an element defined for each pair of commuting elements h, $g \in \pi$, and $\mathrm{Tr}: A \to \hat{\mathbb{Z}}_p$ is the trace map). Finally, we set

$$\mathscr{U}_0(A\pi) = \mathrm{Ker}[v: \mathscr{U}(A\pi) \to H_2(\pi)].$$

The main result is the following (Theorems 3.6 and 4.2 and Proposition 3.8):

THEOREM. *For any prime p, any unramified p-ring A, and any p-group π, there is a natural exact sequence*

$$H_3(\pi) \xrightarrow{\eta_{A\pi}} Wh_2^*(A\pi) \xrightarrow{\Gamma_2^*} \mathscr{U}_0(A\pi) \to 0, \tag{1}$$

with the properties:

(i) $\Gamma_2(\{g, u\}) = g \otimes \Gamma(u)$ *for all* $g \in \pi$ *and* $u \in 1 + I(\hat{\mathbb{Z}}_p[Z_\pi(g)])$ (*where* Γ *is the map defined in* [10] *and I denotes augmentation ideal*).

(ii) $\mathrm{Ker}(\eta_{A\pi})$ *is independent of A.*

(iii) $\eta_{A\pi} = 0$ *if* π *is abelian* (*and so* $Wh_2^*(A\pi) \cong \mathscr{U}_0(A\pi)$).

Furthermore, there is a short exact sequence

$$0 \to H_2(\pi) \to K_2^*(A\pi) \to Wh_2^*(A\pi) \to 0$$

which is naturally split when $A = \hat{\mathbb{Z}}_p$.

The most important point here is the existence of the surjection Γ_2^* having property (i). This is the basis for most of the applications in Sections 5, 6, and 7.

The significance of the contribution of $H_3(\pi)$ to $Wh_2(A\pi)$ is still a mystery. A precise but complicated description of $\mathrm{Ker}(\eta_{A\pi})$ (and also of the extension) is given in Theorem 3.6. This can in particular be used to construct examples for any p of p-groups π where $\eta_{A\pi} \neq 0$ (Proposition 4.3). But we have been unable to find any simple way of describing $\mathrm{Im}(\eta_{A\pi})$.

In particular, these results show that among p-groups π, $K_2^*(\hat{\mathbb{Z}}_p\pi) = 0$ if and only if π is cyclic or quaternionic. That any surjection of a p-group onto $\mathbb{Z}/p^n$ induces a surjection onto $K_2(\hat{\mathbb{Z}}_p[\mathbb{Z}/p^n])$ is well known, but that the corresponding result for quaternionic groups holds seems surprising.

Another interesting consequence comes from combining these results with those in [10]. The image of

$$v: \mathscr{U}(A\pi) \to H_2(\pi)$$

is the subgroup generated by $h \wedge g$ for all commuting h, $g \in \pi$, and this was seen in [10] to be equal to $H_2^{ab}(\pi)$. We can also identify the group $\overline{I(A\pi)}$

(see [10]: $I(A\pi)$ denotes the augmentation ideal) with $H_0(\pi; I(A\pi))$. Sequence (1) above now combines with Theorems 2 and 3 in [10] to give:

THEOREM. *For any unramified p-ring A and any p-group π, there is an exact sequence*

$$H_3(\pi) \xrightarrow{\eta_{A\pi}} Wh_2^*(A\pi) \xrightarrow{\Gamma_2^*} H_1(\pi; I(A\pi))/\langle g \otimes \lambda(g^n - 1)\rangle \to H_2(\pi)$$
$$\longrightarrow Wh(A\pi) \xrightarrow{\Gamma} H_0(\pi; I(A\pi)) \to H_1(\pi) \to 0.$$

An obvious question is whether this can be lifted to some exact sequence involving $Wh_2^{\mathrm{top}}(A\pi)$.

We start in Section 0 with a brief sketch of the construction of sequence (1) above. This is intended to help clarify and make less mysterious the details given later in Sections 2 and 3. Also, at the end of Section 0, we show that the projection $K_2(A\pi) \twoheadrightarrow K_2^*(A\pi)$ factors through

$$K_2^{\mathrm{top}}(A\pi) = \varprojlim_n K_2(A/p^n[\pi]),$$

i.e., that $K_2^*(A\pi)$ can be regarded as a quotient of $K_2^{\mathrm{top}}(A\pi)$.

Section 1 deals with the main technical lemma behind the computation, and this is used in Section 2 to describe $K_1(A\tilde{\pi}, I)$ when $\tilde{\pi} \twoheadrightarrow \pi$ is a surjection of p-groups and $I = \mathrm{Ker}[A\tilde{\pi} \to A\pi]$. The formulas for $K_2^*(A\pi)$ and $Wh_2^*(A\pi)$ are derived in Section 3, and the map $\eta_{A\pi}$ in the above theorems is studied in Section 4.

The last three sections give some applications of these results. In Section 5, the surjectivity of Γ_2^* is combined with localization sequences to get lower bounds on the size of $K_2(\mathbb{Z}\pi)$ and $Wh_2(\pi)$. This extends results of Dennis, Keating, and Stein in [4], results which helped to motivate much of this work.

In Section 6, the analogous groups $K_2^*(A/p^n[\pi])$ are defined, and described in terms of $H_2(\pi)$ and $\mathscr{U}_0(A\pi)$. The main result there is that

$$K_2(A/p[\pi]) = K_2^*(A/p[\pi])$$

for an *abelian* p-group π: this yields an explicit description of $K_2(F\pi)$ when F is any finite field and π any finite abelian group. This result also gives encouragement that $K_2^*(A\pi)$ detects a significant portion of $K_2^{\mathrm{top}}(A\pi)$. The hope, of course, is that the kernel

$$\mathrm{Ker}[K_2^{\mathrm{top}}(A\pi) \twoheadrightarrow K_2^*(A\pi)]$$

will turn out to be generated by an easily describable set of elements. The best possible result would be that it is generated by symbols $\{r, x\}$ for $r \in A^*$ and $x \in (A\pi)^*$, but this is probably too much to hope for.

Finally, in Section 7, applications are given toward the computation of $SK_1(\mathbb{Z}\pi)$ or $Cl_1(\mathbb{Z}\pi)$ for p-groups π. The results on $K_2^*(\hat{\mathbb{Z}}_p\pi)$ are used to get upper and lower bounds for $Cl_1(\mathbb{Z}\pi)$, as well as to motivate a conjecture for a combinatorial algorithm for computing $Cl_1(\mathbb{Z}\pi)$ when π is a p-group and p is odd. As an example, $SK_1(\mathbb{Z}\pi)$ is then computed when π is a non-abelian group of order p^3 (and p any odd prime).

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We start by sketching a quick construction of the exact sequence

$$H_3(\pi) \to Wh_2^*(A\pi) \to \mathscr{U}(A\pi) \xrightarrow{\nu} H_2(\pi).$$

This is much simpler than the construction to be given in detail in Sections 2 and 3, but it gives no information on the size of the image of $H_3(\pi)$ in $Wh_2(A\pi)$, nor on how the extension at $Wh_2^*(A\pi)$ behaves.

Throughout this section, we let A be a fixed unramified p-ring (i.e., the ring of integers in some finite unramified extension of $\hat{\mathbb{Q}}_p$), and let π be a fixed p-group. For any surjection $\alpha: \tilde{\pi} \twoheadrightarrow \pi$ of p-groups, define a group G and an order $\mathfrak{A}$ as pullbacks

$$\begin{array}{ccc} G & \xrightarrow{r_1} & \tilde{\pi} \\ \downarrow{\scriptstyle r_2} & (1) & \downarrow{\scriptstyle \alpha} \\ \tilde{\pi} & \xrightarrow{\alpha} & \pi \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \mathfrak{A} & \xrightarrow{r_1} & A\tilde{\pi} \\ \downarrow{\scriptstyle r_2} & (2) & \downarrow \\ A\tilde{\pi} & \longrightarrow & A\pi \end{array}$$

We regard G, $\mathfrak{A}$, and $\tilde{\pi}$ as depending on α. Set

$$I_i = \operatorname{Ker}[r_i: AG \to A\tilde{\pi}] \qquad (i = 1, 2)$$

and let $\psi: AG \twoheadrightarrow \mathfrak{A}$ be the obvious surjection. Then $\operatorname{Ker}(\psi) = I_1 \cap I_2 = I_1 I_2$ (see Lemma 2.4 below); so $\mathfrak{A} \cong AG/I_1 I_2$, and

$$K_1(\mathfrak{A}) \cong K_1(AG)/(1 + I_1 I_2). \tag{3}$$

The five-term homology sequences for the extensions

$$1 \to \rho_1 \to G \xrightarrow{r_1} \tilde{\pi} \to 1 \qquad \text{and} \qquad 1 \to \rho \to \tilde{\pi} \xrightarrow{\alpha} \pi \to 1$$

(see Proposition 2.1) induce a Mayer–Vietoris exact sequence

$$H_2(\tilde{\pi}) \oplus H_2(\tilde{\pi}) \to H_2(\pi) \to G^{ab} \to \tilde{\pi}^{ab} \oplus \tilde{\pi}^{ab} \to \pi^{ab} \to 1 \tag{4}$$

(note that $\rho_1 \cong \rho$ and $\rho_1/[G, \rho_1] \cong \rho/[\tilde{\pi}, \rho]$). Upon comparing this with the K-theory exact sequence for (2), we get an exact sequence of Whitehead groups:

$$Wh_2(A\tilde{\pi}) \oplus Wh_2(A\tilde{\pi}) \longrightarrow Wh_2(A\pi) \xrightarrow{\partial} \frac{Wh(AG)}{1+I_1 I_2} \xrightarrow{r_{1*} \oplus r_{2*}} Wh(A\tilde{\pi}) \oplus Wh(A\tilde{\pi}). \tag{5}$$

In other words,

$$\begin{aligned} &\operatorname{Coker}[\alpha_*: Wh_2(A\tilde{\pi}) \to Wh_2(A\pi)] \\ &\cong \operatorname{Ker}\left[r_{1*} \oplus r_{2*}: \frac{Wh(AG)}{1+I_1 I_2} \to Wh(A\tilde{\pi})^2 \right]; \end{aligned}$$

and in the limit

$$Wh_2^*(A\pi) \cong \varprojlim_{\alpha} \operatorname{Ker}\left[r_{1*} \oplus r_{2*}: \frac{Wh(AG)}{1+I_1 I_2} \to Wh(A\tilde{\pi})^2 \right]. \tag{6}$$

At this point, the modified logarithm map studied in Section 2 of [10] can be applied. Let

$$I(A\tilde{\pi}) = \operatorname{Ker}[\varepsilon: A\tilde{\pi} \to A] \qquad \text{and} \qquad \overline{I(A\tilde{\pi})} = H_0(\tilde{\pi}; I(A\tilde{\pi})).$$

Here, ε is the augmentation map, and $\tilde{\pi}$ acts on $I(A\tilde{\pi})$ by conjugation. By Theorem 2 in [10], there is a map $\Gamma = \Gamma_{A\pi}$ and a short exact sequence

$$0 \to Wh'(A\tilde{\pi}) \xrightarrow{\Gamma_{A\pi}} \overline{I(A\tilde{\pi})} \xrightarrow{\omega} \tilde{\pi}^{ab} \to 0$$

(and similarly for G), where $\omega(\sum \lambda_i g_i) = \prod g_i^{\operatorname{Tr}(\lambda_i)}$ and

$$Wh'(A\pi) = Wh(A\pi)/SK_1(A\pi).$$

The main result of Section 1 below (Theorem 1.1) says that

$$\Gamma_G(1 + I_1 I_2) = \overline{I_1 I_2},$$

where $\overline{I_1 I_2}$ denotes the image of $I_1 I_2$ in $\overline{I(AG)}$. From this we get

$$\begin{aligned} &\operatorname{Ker}\left[r_1 \oplus r_2: \frac{Wh'(AG)}{1+I_1 I_2} \to Wh'(A\tilde{\pi})^2 \right] \\ &\cong \operatorname{Ker}\left[(\omega, r_1, r_2): H_0\left(G; \frac{I(AG)}{I_1 I_2} \right) \to G^{ab} \oplus H_0(\tilde{\pi}; I(A\tilde{\pi}))^2 \right] \\ &\cong \operatorname{Ker}[(\omega, i_*): H_0(G; I_\alpha) \to G^{ab} \oplus H_0(\tilde{\pi}; I(A\tilde{\pi}))]. \end{aligned} \tag{7}$$

Here,

$$I_\alpha = \operatorname{Ker}[\alpha\colon A\tilde{\pi} \to A\pi] \cong \operatorname{Ker}[r_1\colon \mathfrak{A} \to A\tilde{\pi}]$$
$$\cong \operatorname{Ker}\left[r_1\colon \frac{I(AG)}{I_1 I_2} \to I(A\tilde{\pi})\right]$$

(in particular, the G-action on I_α factors through $\tilde{\pi}$). Recall that r_1 is split by the diagonal map $\Delta\colon A\tilde{\pi} \to \mathfrak{A}(\subseteq (A\tilde{\pi})^2)$.

In Proposition 3.3 and Lemma 3.4, we will see that

$$\varprojlim_\alpha \operatorname{Ker}[i_*\colon H_0(\tilde{\pi}; I_\alpha) \to H_0(\tilde{\pi}; I(A\tilde{\pi}))]$$
$$\cong \varprojlim_\alpha \operatorname{Coker}[\alpha_*\colon H_1(\tilde{\pi}; I(A\tilde{\pi})) \to H_1(\tilde{\pi}; I(A\pi))]$$
$$\cong H_1(\pi, A\pi)/\langle g \otimes \lambda g^n\colon g \in \pi, \lambda \in A, n \in \mathbb{Z}\rangle$$
$$= \mathscr{U}(A\pi).$$

The main point here is that if $g, h \in \pi$ and $\langle g, h\rangle$ is not cyclic, then there is some $\alpha\colon \tilde{\pi} \twoheadrightarrow \pi$ such that no liftings of g and h commute in $\tilde{\pi}$.

So (7) now takes the form

$$\varprojlim_\alpha \operatorname{Ker}\left[r_{1*} \oplus r_{2*}\colon \frac{Wh'(AG)}{1 + I_1 I_2} \to Wh'(A\tilde{\pi})^2\right]$$
$$\cong \operatorname{Ker}\left[\mathscr{U}(A\pi) \xrightarrow{\omega|} \varprojlim_\alpha \operatorname{Ker}[G^{ab} \to \tilde{\pi}^{ab} \oplus \tilde{\pi}^{ab}] \cong H_2(\pi)\right]. \tag{8}$$

That the last limit is isomorphic to $H_2(\pi)$ follows from sequence (4). The composite of these maps is checked to be the map

$$\nu\colon \mathscr{U}(A\pi) \to H_2(\pi),$$

where $(g \otimes \lambda h) = \operatorname{Tr}(\lambda) \cdot (g \wedge h)$ (in the notation of Proposition 2.1 below).

It remains to see how $SK_1(A\tilde{\pi})$ and $SK_1(AG)$ behave in the limit. That $\varprojlim_\alpha SK_1(A\tilde{\pi}) = 0$ follows easily from Lemma 22(i) in [10]. Spectral sequences show that

$$\varprojlim_\alpha SK_1(AG) \cong \varprojlim_\alpha (H_2(G)/H_2^{ab}(G)) \cong H_3(\pi)/H_3^{cy}(\pi), \tag{9}$$

where $H_3^{cy}(\pi)$ is the subgroup of $H_3(\pi)$ generated by induction from cyclic subgroups. So upon combining (6), (8), and (9), we get:

THEOREM. *For any p-group π and any unramified p-ring A, there is an exact sequence*

$$H_3(\pi) \xrightarrow{\eta_{A\pi}} Wh_2^*(A\pi) \to \mathscr{U}(A\pi) \xrightarrow{\nu} H_2(\pi).$$

Describing $\mathrm{Ker}(\eta_{A\pi})$ amounts to computing, for each $\alpha: \tilde{\pi} \twoheadrightarrow \pi$, how much of $SK_1(AG)$ is hit by $(1+I_1I_2)$. To do this, better control is needed over $Wh(AG)$ as an extension of $SK_1(AG)$ by $Wh'(AG)$. This is obtained by regarding $Wh(AG)$ as a quotient group of $Wh'(A\hat{G})$ (and hence as a subquotient of $\overline{I(A\hat{G}))}$ for some central extension $\hat{G} \twoheadrightarrow G$ for which $SK_1(A\hat{G})=0$ (such G exists by [10, Lemma 22(i)]). The extra complications in the constructions in Sections 2 and 3 below are caused precisely by the need to work with $\hat{G}$ instead of G.

We end this introductory section by noting that $K_2^*(A\pi)$ can be regarded as a quotient of $K_2^{\mathrm{top}}(A\pi)$, as well as of $K_2(A\pi)$ (that $K_2(A\pi)$ surjects onto $K_2^*(A\pi)$ is clear once we know that $K_2^*(A\pi)$ is finite). As usual, for any p-adic order $\mathfrak{A}$, we set

$$K_2^{\mathrm{top}}(\mathfrak{A}) = \varprojlim_n K_2(\mathfrak{A}/p^n(\mathfrak{A}).$$

PROPOSITION 0.1. *For any p-group π and any unramified p-ring A,*

$$K_2^*(A\pi) \cong \varprojlim \mathrm{Coker}[K_2^{\mathrm{top}}(A\tilde{\pi}) \to K_2^{\mathrm{top}}(A\pi)].$$

In particular, the natural projection $[K_2(A\pi) \twoheadrightarrow K_2^*(A\pi)]$ factors through $K_2^{\mathrm{top}}(A\pi)$.

Proof. Let $\alpha: \tilde{\pi} \to \pi$ be any surjection of a p-group $\tilde{\pi}$ onto π, and define pullback rings $\mathfrak{A}_n$ $(n \geqslant 1)$ and $\mathfrak{A}$ from the squares

$$\begin{array}{ccc} \mathfrak{A}_n & \longrightarrow & A/p^n[\tilde{\pi}] \\ \downarrow & (1) & \downarrow \\ A/p^n[\tilde{\pi}] & \longrightarrow & A/p^n[\pi] \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \mathfrak{A} & \longrightarrow & A\tilde{\pi} \\ \downarrow & (2) & \downarrow \\ A\tilde{\pi} & \longrightarrow & A\pi \end{array}$$

Then $\mathfrak{A}_n \cong \mathfrak{A}/p^n\mathfrak{A}$. So by Theorem 10 (and its proof) in [14],

$$K_1(\mathfrak{A}) \cong \varprojlim_n K_1(\mathfrak{A}_n) \qquad \text{and} \qquad K_1(A\tilde{\pi}) \cong \varprojlim_n K_1(A/p^n[\tilde{\pi}]).$$

The Mayer–Vietoris sequences for (1) and (2) thus induce isomorphisms

$$\begin{aligned} \mathrm{Coker}[K_2^{\mathrm{top}}(A\tilde{\pi}) \to K_2^{\mathrm{top}}(A\pi)] &\cong \mathrm{Ker}[K_1(\mathfrak{A}) \to K_1(A\tilde{\pi}) \oplus K_1(A\tilde{\pi})] \\ &\cong \mathrm{Coker}[K_2(A\tilde{\pi}) \to K_2(A\pi)]. \end{aligned}$$

Hence, in the limit,

$$K_2^*(A\pi) \cong \varprojlim_{\tilde{\pi}} \mathrm{Coker}[K_2^{\mathrm{top}}(A\tilde{\pi}) \to K_2^{\mathrm{top}}(A\pi)]. \quad \blacksquare$$

1

We start with some general definitions. If G is any group and R any commutative ring, set

$$\overline{RG} = RG/\langle \lambda(g - hgh^{-1}) \mid \lambda \in R;\ g,\ h \in G \rangle,$$

i.e., the free R-module on the conjugacy classes of G. Let $I(RG)$ denote the augmentation ideal, and

$$\overline{I(RG)} = I(RG)/\langle \text{conjugation} \rangle \subseteq \overline{RG}.$$

If π is a p-group and A an unramified p-ring, let

$$\Gamma_{A\pi}\colon Wh(A\pi) \to \overline{I(A\pi)}$$

be the map studied in Section 2 of [10]. The main result of this section is:

THEOREM 1.1 *Let π be a p-group, and let $\sigma, \tau \lhd \pi$ be normal subgroups such that $[\sigma, \tau] = 1$. Let A be an unramified p-ring, and set*

$$I_{A\sigma} = \operatorname{Ker}[A\pi \to A[\pi/\sigma]] \qquad \text{and} \qquad I_{A\tau} = \operatorname{Ker}[A\pi \to A[\pi/\tau]].$$

Then

$$\Gamma_{A\pi}(1 + I_{A\sigma} I_{A\tau}) = \overline{I_{A\sigma} I_{A\tau}},$$

where $\overline{I_{A\sigma} I_{A\tau}}$ denotes the image of $I_{A\sigma} I_{A\tau}$ in $\overline{I(A\pi)}$.

Before Theorem 1.1 can be proven, some quite technical lemmas will be needed. The first one looks rather strange, but seems to appear frequently in computations involving $\Gamma_{A\pi}$ (see also the proof in [10] that $\Gamma_{A\pi}$ is integer-valued).

LEMMA 1.2. *Let p be a prime, A a commutative ring, and $\varphi \in \operatorname{Aut}(A)$ a ring homomorphism such that $\varphi(\lambda) \equiv \lambda^p \pmod{p}$ for $\lambda \in A$. Let R be an A-algebra, $U \subseteq R$ a multiplicative set, and $Y \subseteq R$ an A-submodule containing U. Set*

$$S = \left\{ \sum_i (x_i y_i - y_i x_i) \mid x_i,\ y_i \in R \right\}.$$

Let $\Phi\colon R \to R$ be any φ-linear module homomorphism such that

$$x^{p^n} - \Phi(x^{p^{n-1}}) \in S + p^n Y \tag{1}$$

for all $x \in U$ and $n \geqslant 1$. Then (1) *holds for all $x = \sum_i \lambda_i x_i$, where $\lambda_i \in A$ and $x_i \in U$ (and all $n \geqslant 1$).*

Proof. Fix some $x = \sum_i \lambda_i x_i$, where $\lambda_i \in A$ and $x_i \in U$. A typical term in x^{p^n} has the form

$$\xi = \lambda_{i_1} \cdots \lambda_{i_q} x_{i_1} \cdots x_{i_q} \qquad (q = p^n).$$

Cyclic permutation of the indices $(i_1, \ldots, i_q)$ give a total of p^m terms $\xi_1, \ldots, \xi_{p^m}$ congruent to ξ (mod S), where p^{n-m} is the number of cyclic permutations leaving $(i_1, \ldots, i_q)$ invariant. Then $x_{i_1} \cdots x_{i_q}$ is a p^{n-m}th power, and the sum of these terms (mod S) has the form

$$\sum_{i=1}^{p^m} \xi_i \equiv \eta = p^m \bar{\lambda}^{p^{n-m}} \bar{x}^{p^{n-m}} \qquad (\bar{\lambda} = \lambda_{i_1} \cdots \lambda_{i_r},\ \bar{x} = x_{i_1} \cdots x_{i_r},\ r = p^m).$$

If $m = n$, then $\eta \in p^n Y$ by assumption. If $m < n$, then there are corresponding terms

$$\sum_{i=1}^{p^m} \xi'_1 \equiv \eta' = p^m \varphi(\bar{\lambda})^{p^{n-m-1}} \Phi(\bar{x}^{p^{n-m-1}})$$

in the expansion of $\Phi(x^{p^{n-1}})$. Since $\varphi(\bar{\lambda}) \equiv \bar{\lambda}^p$ (mod p) by assumption,

$$\varphi(\bar{\lambda})^{p^{n-m-1}} \equiv \bar{\lambda}^{p^{n-m}} \qquad (\mathrm{mod}\ p^{n-m}).$$

Also, $\bar{x}^{p^{n-m}} \in Y$ and $\bar{x}^{p^{n-m}} - \Phi(\bar{x}^{p^{n-m-1}}) \in S + p^{n-m} Y$ $(\bar{x} \in U)$, so

$$\eta - \eta' \equiv p^m [\bar{\lambda}^{p^{n-m}} \bar{x}^{p^{n-m}} - \varphi(\bar{\lambda})^{p^{n-m-1}} \Phi(\bar{x}^{p^{n-m-1}})] \in S + p^n Y.$$

The result follows by summing over all such (η, η'). ∎

For Lemmas 1.3 and 1.4, we fix $s, t \geqslant 1$, and let F be the free group on generators $h_1, \ldots, h_s, k_1, \ldots, k_t, g$. Let $\tilde{H}$ and $\tilde{K}$ be the normal subgroups of F generated by the h_i and k_i, respectively. Set

$$G = F/[\tilde{H}, \tilde{K}], \qquad H = \tilde{H}/[\tilde{H}, \tilde{K}], \qquad K = \tilde{K}/[\tilde{H}, \tilde{K}],$$

and let $H_0 = [H, G]$ and $K_0 = [K, G]$. Define

$$I_H = \mathrm{Ker}[\mathbb{Z}G \to \mathbb{Z}[G/H]], \qquad I_K = \mathrm{Ker}[\mathbb{Z}G \to \mathbb{Z}[G/K]],$$

and similarly for I_{H_0} and I_{K_0}. Finally, set

$$S = \mathrm{Ker}[\mathbb{Z}G \to \overline{\mathbb{Z}G}],$$

i.e., the subgroup of $\mathbb{Z}G$ generated by all $u - vuv^{-1}$ for $u, v \in G$.

LEMMA 1.3. *For any $u \in G$, let $\delta(u)$ be the unique element of the form*

$$\delta(u) = h_1^{a_1} \cdots h_s^{a_s} k_1^{b_1} \cdots k_t^{b_t} g^c$$

such that $u^{-1}\delta(u) \in [G, G]$. Let $\delta: \mathbb{Z}G \to \mathbb{Z}G$ *be the induced ($\hat{\mathbb{Z}}_p$-linear) homomorphism. Then, for any*

$$x \in (S + I_H) \cap (S + I_K),$$

$$x \equiv \delta(x) \qquad (\mathrm{mod}\ I_H I_{K_0} + I_{H_0} I_K + S).$$

Proof. We may clearly assume $x \in I_K$.

Step 1. For any $u \in G$, let $\delta_K(u) \in G$ be the unique element of the form

$$\delta_K(u) = h_1^{a_1} \cdots h_s^{a_s} u' \qquad (u' \in \langle k_1, ..., k_t, g \rangle)$$

such that $u^{-1}\delta_K(u) \in H_0 = [H, G]$. Let $\delta_K: \mathbb{Z}G \to \mathbb{Z}G$ be the induced homomorphism. Write

$$x = \sum_i \lambda_i u_i(v_i - 1) \qquad (\lambda_i \in \mathbb{Z},\ u_i \in G,\ v_i \in K);$$

then $\delta_K(u_i v_i) = \delta_K(u_i) \cdot v_i$ ($v_i \in K \subseteq \langle k_1, ..., k_t, g \rangle$, since $[H, K] = 1$). So

$$\delta_K(x) = \sum_i \lambda_i \delta_K(u_i)(v_i - 1) \equiv \sum_i \lambda_i u_i(v_i - 1) = x \qquad (\mathrm{mod}\ I_{H_0} I_K). \tag{1}$$

Step 2. Now write

$$\delta_K(x) = \sum_{i=1}^{r} \lambda_i u_i v_i, \tag{2}$$

where $\lambda_i \in \mathbb{Z}$, $u_i = h_1^{a_{i1}} \cdots h_s^{a_{is}}$, and $v_i \in \langle k_1, ..., k_t, g \rangle$. By assumption,

$$\delta_K(x) \in I_H + S = \mathrm{Ker}[\mathbb{Z}G \to \overline{\mathbb{Z}[G/H]}].$$

In other words, since $\langle k_1, ..., k_t, g \rangle$ maps isomorphically onto G/H, there are elements $\gamma_i \in \langle k_1, ..., k_t, g \rangle$ such that

$$\sum_i \lambda_i \cdot \gamma_i v_i \gamma_i^{-1} = 0.$$

Then

$$\delta_K(x) \equiv x_1 = \sum_{i=1}^{r} \lambda_i (\gamma_i u_i \gamma_i^{-1})(\gamma_i v_i \gamma_i^{-1}) \qquad (\mathrm{mod}\ S), \tag{3}$$

and $x_1 \in I_H$. Since

$$\delta(v_i) = \delta(\gamma_i v_i \gamma_i^{-1}) \equiv \gamma_i v_i \gamma_i^{-1} \qquad (\mathrm{mod}\ K_0 = [K, G]),$$

we get in the same way as in Step 1 that

$$x_1 \equiv x_2 = \sum_{i=1}^{r} \lambda_i(\gamma_i u_i \gamma_i^{-1})\, \delta(v_i) \qquad (\mathrm{mod}\ I_H I_{K_0}). \tag{4}$$

Step 3. Referring to (2), we see that

$$\delta(x) = \delta(\delta_K(x)) = \sum_i \lambda_i u_i \delta(v_i).$$

Furthermore, since $\gamma_i \in \langle k_1,..., k_t, g\rangle$ and $[H, K] = 1$,

$$\gamma_i u_i \gamma_i^{-1} = g^{d_i} u_i g^{-d_i}$$

for some d_i. By (1), (3), and (4),

$$x \equiv x_2 = \sum_{i=1}^{r} \lambda_i(g^{d_i} u_i g^{-d_i})\, \delta(v_i) \qquad (\mathrm{mod}\ I_H I_{K_0} + I_{H_0} I_K + S),$$

and it remains to show that

$$(g^d h g^{-d})\, k g^c \equiv h k g^c \qquad (\mathrm{mod}\ I_{H_0} I_K + S)$$

for any $h \in \langle h_1,..., h_s\rangle$, $k \in \langle k_1,..., k_t\rangle$, and $c, d \in \mathbb{Z}$. But

$$\begin{aligned} hkg^c &= h(kg^c - g^c) + hg^c \\ &\equiv g^d h g^{-d}(kg^c - g^c) + hg^c \qquad (\mathrm{mod}\ I_{H_0} I_K) \\ &= (g^d h g^{-d})\, kg^c - g^d h g^{-d} g^c + hg^c \\ &\equiv (g^d h g^{-d})\, kg^c \qquad (\mathrm{mod}\ S). \quad \blacksquare \end{aligned}$$

LEMMA 1.4. *In the situation of Lemma* 1.3, *set*

$$\xi = (1 - h_1) \cdots (1 - h_s)(1 - k_1) \cdots (1 - k_t)\, g \in \mathbb{Z}G.$$

Fix a prime p, and define $\Phi: \mathbb{Z}G \to \mathbb{Z}G$ by

$$\Phi\left(\sum \lambda_i g_i\right) = \sum \lambda_i g_i^p \qquad (\lambda_i \in \mathbb{Z},\ g_i \in G).$$

Set $H_1 = \langle [H, G], H^p\rangle \lhd G$. Then for any $n \geqslant 1$,

$$\xi^{p^n} - \Phi(\xi^{p^{n-1}}) \in S + p^n I_{H_1} I_K + p^n I_H I_{K_0} + p^n I_H^s I_K^{t+1}.$$

Proof. We first check that

$$\xi^{p^n} - \Phi(\xi^{p^{n-1}}) \in S + p^n \mathbb{Z}G. \tag{1}$$

Apply Lemma 1.2, with $R=Y=\mathbb{Z}G$, $U=G$, $A=\mathbb{Z}$, and $\varphi=id$. Note that

$$S=\left\{\sum (x_i y_i - y_i x_i) \mid x_i,\ y_i \in \mathbb{Z}G\right\},$$

since the group elements generate $\mathbb{Z}G$ additively. Since $u^{p^n}=\Phi(u^{p^{n-1}})$ for $u\in G$, by definition, (1) follows by Lemma 1.2.

Now fix $x\in\mathbb{Z}G$ such that $\xi^{p^n}-\Phi(\xi^{p^{n-1}})\equiv p^n x \pmod S$. Then x vanishes in $\overline{\mathbb{Z}[G/H]}$ and $\overline{\mathbb{Z}[G/K]}$: they are torsion free, and $\xi\in I_H\cap I_K$. In other words,

$$x\in(S+I_H)\cap(S+I_K),$$

and in the notation of Lemma 1.3,

$$x\equiv\delta(x) \qquad (\mathrm{mod}\ I_H I_{K_0}+I_{H_0}I_K+S). \tag{2}$$

Since $\delta(S)=0$,

$$\begin{aligned}\delta(x)&=\frac{1}{p^n}\,[\delta(\xi^{p^n})-\delta(\Phi(\xi^{p^{n-1}}))]\\&=\frac{1}{p^n}\,[(1-h_1)^{p^n}\cdots(1-h_s)^{p^n}\cdots(1-k_t)^{p^n}g^{p^n}\\&\quad-(1-h_1^p)^{p^{n-1}}\cdots(1-k_t^p)^{p^{n-1}}g^{p^n}].\end{aligned} \tag{3}$$

Furthermore,

$$(1-h_i)^{p^n}-(1-h_i^p)^{p^{n-1}}\in p^n I_H$$

and similarly for k_i ($p^n\mathbb{Z}G\cap I_H=p^n I_H$, since $\mathbb{Z}[G/H]$ is torsion free). So (3) implies that

$$\delta(x)\in I_H^s I_K^{t+1}+I_{H_1}I_K. \tag{4}$$

The lemma now follows from (2), (4), and the definition of x. ∎

We now return to the situation of Theorem 1.1. For the rest of Section 1, we fix an unramified p-ring A, a p-group π, and subgroups $\sigma,\tau\lhd\pi$ such that $[\sigma,\tau]=1$. Set

$$\sigma_1=\langle[\sigma,\pi],\mathrm{Fr}(\sigma)\rangle \qquad \text{and} \qquad \tau_1=\langle[\tau,\pi],\mathrm{Fr}(\tau)\rangle,$$

where $\mathrm{Fr}(\sigma)$ and $\mathrm{Fr}(\tau)$ denote the Frattini subgroups. Note that $\sigma_1,\tau_1\lhd\pi$, and $\sigma_1\subsetneqq\sigma$, $\tau_1\subsetneqq\tau$. Hence, upon setting

$$X=I_{A\sigma_1}I_{A\tau}+I_{A\sigma}I_{A\tau_1}\subseteq A\pi$$

($I_{\sigma_1} = \mathrm{Ker}[A\pi \to A[\pi/\sigma_1]]$, etc.), we may assume inductively that $\Gamma_{A\pi}(1+X) = \bar{X}$ (the image of X in $\overline{I(A\pi)}$).

Recall that $\Gamma_{A\pi}$ is defined by setting

$$\Gamma_{A\pi}(x) = \log x - \frac{1}{p}\,\Phi(\log x) \in \overline{I(A\pi)}$$

for any $x \in 1 + I(A\pi)$, where $\Phi(\sum \lambda_i g_i) = \sum \varphi(\lambda_i)\, g_i^p$ and $\varphi \in \mathrm{Gal}(A/\hat{\mathbb{Z}}_p)$ is the Frobenius automorphism ($\varphi(\lambda) \equiv \lambda^p \pmod p$).

LEMMA 1.5. *For any $u, v \geqslant 1$ and $x \in I^u_{A\sigma} I^v_{A\tau}$,*

$$\Gamma_{A\pi}(1+x) \equiv x \qquad (\mathrm{mod}\ I^u_{A\sigma} I^{v+1}_{A\tau} + X) \text{ in } \overline{I(A\pi)}.$$

Proof. First note that

$$\begin{aligned}\Gamma(1+x) &= \left[x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \cdots\right] - \left[\frac{1}{p}\,\Phi(x) - \frac{1}{2p}\,\Phi(x^2) + \cdots\right]\\ &\equiv x + \sum_{k=1}^{\infty} (-1)^{k+1}\frac{1}{pk}\,[x^{pk} - \Phi(x^k)] \qquad (\mathrm{mod}\ I^{2u}_{A\sigma} I^{2v}_{A\tau}).\end{aligned}$$

(Special care is needed when $p = 2$.) So it suffices to show that

$$x^{p^n} - \Phi(x^{p^{n-1}}) \in S' + p^n(X + I^u_{A\sigma} I^{v+1}_{A\tau}) \tag{1}$$

for all $n \geqslant 1$ and $x \in I^u_{A\sigma} I^v_{A\tau}$, where

$$S' = \mathrm{Ker}[I(A\pi) \to \overline{I(A\pi)}] = \left\{\sum_i (x_i y_i - y_i x_i) \mid x_i, y_i \in A\pi\right\}.$$

We again apply Lemma 1.2, this time with $R = A\pi$,

$$\begin{aligned}U = \{&(1-h_1)\cdots(1-h_s)(1-k_1)\cdots(1-k_t)\, g \mid h_i \in \sigma, k_i \in \tau,\\ &g \in \pi, s \geqslant u, t \geqslant v\},\end{aligned}$$

and $Y = X + I^u_{A\sigma} I^{v+1}_{A\tau}$. Since $[\sigma, \tau] = 1$, U is multiplicative. Relation (1) holds for all $x \in U$ by Lemma 1.4. So by Lemma 1.2, (1) holds for all

$$x \in I^u_{A\sigma} I^v_{A\tau} = \left\{\sum_i \lambda_i x_i \mid \lambda_i \in A, x_i \in U\right\}. \quad \blacksquare$$

Poof of Theorem 1.1. First note that $\Gamma(1 + I_{A\sigma} I_{A\tau}) \subseteq \overline{I_{A\sigma} I_{A\tau}}$ by Lemma 1.5. As remarked, $\Gamma(1+X) = \bar{X}$ by induction, and hence

$$\bar{X} \subseteq \Gamma(1 + I_{A\sigma} A_{A\tau}).$$

We next check that $\Phi(x) \in X$ for all $x \in I_{A\sigma} I_{A\tau}$. It suffices to show this when $x = (1-h)(1-k)\, g$ for any $h \in \sigma$, $k \in \tau$, and $g \in \pi$. For such x,

$$\Phi(x) = g^p - (hg)^p - (kg)^p + (hkg)^p = (1-h')(1-k')\, g,$$

where

$$h' = h(ghg^{-1}) \cdots (g^{p-1}hg^{-p+1}) \qquad \text{and} \qquad k' = k(gkg^{-1}) \cdots (g^{p-1}kg^{-p+1})$$

(recall that $[\sigma, \tau] = 1$). Furthermore,

$$h' \in \sigma_1 = \langle [\sigma, \pi], \mathrm{Fr}(\sigma) \rangle \qquad \text{and} \qquad k' \in \tau_1 = \langle [\tau, \pi], \mathrm{Fr}(\tau) \rangle,$$

and so $\Phi(x) \in I_{A\sigma_1} I_{A\tau_1} \subseteq X$.

Thus, for any $n \geqslant 2$ and any $x \in I_{A\sigma} I_{A\tau}$,

$$\begin{aligned} \Gamma(1 + p^n x) &\equiv \log(1 + p^n x) && (\mathrm{mod}\ \bar{X}) \\ &\equiv p^n x && (\mathrm{mod}\ \bar{X} + p^{n+1} \overline{I_{A\sigma} I_{A\tau}}). \end{aligned}$$

By taking successive approximations, it follows that

$$\Gamma(1 + I_{A\sigma} I_{A\tau}) \supseteq p^2 I_{A\sigma} I_{A\tau}.$$

Finally, note that for t large enough,

$$I^t_{A\tau} \subseteq p^2 I_{A\tau} \qquad \text{and} \qquad I_{A\sigma} I^t_{A\tau} \subseteq p^2 I_{A\sigma} I_{A\tau}.$$

So by Lemma 1.5 and downward induction,

$$\Gamma(1 + I_{A\sigma} I_{A\tau}) \supseteq I_{A\sigma} I^t_{A\tau}$$

for all $t \geqslant 1$. In particular, $\Gamma(1 + I_{A\sigma} I_{A\tau}) = \overline{I_{A\sigma} I_{A\tau}}$. ∎

2

Throughout this section, A will be a fixed unramified p-ring (for some fixed prime p), and $\varphi \in \mathrm{Gal}(A/\hat{\mathbb{Z}}_p)$ the Frobenius automorphism. We let $\mathrm{Tr} = \mathrm{Tr}_{A/\hat{\mathbb{Z}}_p}$ denote the trace map; it is surjective by Proposition VIII.1.3 in [15]. The goal is to describe $K_1(A\tilde{\pi}, I)$ when $\tilde{\pi}$ is a p-group and $I = \mathrm{Ker}[A\tilde{\pi} \to A\pi]$ for some surjection $\tilde{\pi} \to \pi$.

We first list some basic facts involving $H_2(\pi)$ which will be needed frequently.

PROPOSITION 2.1. (i) *For any extension* $1 \to \rho \to \pi \to^{\alpha} \pi \to 1$ *of groups, there is a 5-term homology exact sequence natural in* α:

$$H_2(\tilde{\pi}) \xrightarrow{\alpha_*} H_2(\pi) \xrightarrow{\delta^\alpha} \rho/[\tilde{\pi}, \rho] \to \tilde{\pi}^{ab} \to \pi^{ab} \to 0.$$

(ii) *If* $1 \to R \to F \to^{\alpha} \pi \to 1$ *is an extension, where* F *is free and* $\exp H_2(\pi) \mid q < \infty$, *then the composite*

$$H_2(\pi) \xrightarrow{\delta^\alpha} R/[F, R] \twoheadrightarrow (R/[F, R]) \otimes \mathbb{Z}/q$$

is injective.

(iii) *If* $\pi = F/R$ *where* F *is free, then* $H_2(\pi) \simeq (R \cap [F, F])/[R, F]$. *In particular, if* $g, h \in \pi$ *and* $gh = hg$, *then they determine an element* $g \wedge h \in H_2(\pi)$: $g \wedge h = [\tilde{g}, \tilde{h}] \in R \cap [F, F]$ *for any liftings* $\tilde{g}, \tilde{h} \in F$ *of* g *and* h. *We define*

$$H_2^{ab} = \langle g \wedge h \in H_2(\pi) : g, h \in \pi, gh = hg \rangle \subseteq H_2(\pi).$$

(iv) *For any p-group* π,

$$SK_1(A\pi) \cong H_2(\pi)/H_2^{ab}(\pi).$$

For any central *extension* $1 \to \rho \to \tilde{\pi} \to^{\alpha} \pi \to 1$ *of p-groups,*

$$SK_1(A\tilde{\pi}) \cong \operatorname{Ker}(\delta^\alpha)/\langle g \wedge h : g, h \in \pi, gh = hg, g \wedge h \in \operatorname{Ker}(\delta^\alpha) \rangle.$$

In particular, $SK_1(A\tilde{\pi}) \cong SK_1(A\pi)$ *if* $\delta^\alpha = 0$, *and* $SK_1(A\tilde{\pi}) = 0$ *if* δ^α *is injective.*

Proof. The homology sequence is constructed, e.g., in [6, Corollary VI.8.2]. To see (ii), note that $\operatorname{Coker}(\delta^\alpha) \subseteq F^{ab}$ is free abelian and $H_2(F) = 0$, so that δ^α is split injective. For (iv), see Theorem 3 and Lemma 22(i) in [10]. ∎

From now on, for any extension $1 \to \rho \to \tilde{\pi} \to^{\alpha} \pi \to 1$ of groups, we set

$$I_{A\alpha} = \operatorname{Ker}[A\tilde{\pi} \xrightarrow{\alpha} A\pi].$$

Maps

$$\omega^* = \omega^*_{A\alpha} : I_{A\alpha} \to A \otimes \rho^{ab} \qquad \text{and} \qquad \omega = \omega_{A\alpha} : I_{A\alpha} \to \hat{\mathbb{Z}}_p \otimes \rho^{ab}$$

are defined by setting

$$\omega^*\left(\sum_i \lambda_i(h_i - 1)\, g_i\right) = \sum_i \lambda_i \otimes h_i \qquad (\lambda_i \in A, h_i \in \rho, g_i \in \tilde{\pi}),$$

and letting $\omega = (\operatorname{Tr} \otimes 1)\, \omega^*$. Note that $\hat{\mathbb{Z}}_p \otimes \rho^{ab} \cong \rho^{ab}$ if ρ is a p-group.

PROPOSITION 2.2. *Let* $1 \to \rho \to \tilde{\pi} \to^{\alpha} \pi \to 1$ *be an extension of groups.*

(i) $\mathrm{Ker}(\omega^*_{A\alpha}) = I_{A\alpha} I$, *where* $I = I(A\tilde{\pi})$. *In other words,* $\omega^*_{A\alpha}$ *induces an isomorphism.*

$$I_{A\alpha}/I_{A\alpha} I \cong A \otimes \rho^{ab}.$$

(ii) *If* $\tilde{\pi}$ *is a p-group and* $\rho \subseteq Z(\tilde{\pi})$ *is central, then*

$$\Gamma_{A\tilde{\pi}}(1 + I_{A\alpha}) = \mathrm{Ker}[\omega_{A\alpha}: \bar{I}_{A\alpha} \to \rho] \subseteq \overline{I(A\tilde{\pi})}.$$

Proof. (i) Define

$$s: A \otimes \rho^{ab} \to I_{A\alpha}/I_{A\alpha} I$$

by setting $s(\lambda \otimes h) = \lambda(h-1)$. This is clearly a homomorphism:

$$s(\lambda \otimes hk) - s(\lambda \otimes h) - s(\lambda \otimes k) = \lambda(1-h)(1-k) \in (I_{A\alpha})^2$$

for any $\lambda \in A$ and $h, k \in p$. That $\omega^*_{A\alpha}$ and s are inverses is easy.

(ii) By (i), $I_{A\alpha}$ is generated by $I_{A\alpha} I$ and $I(A\rho)$ (the augmentation ideal in $A\rho$). Hence, by Theorem 1.1 (applied to the pair $(\sigma, \tau) = (\rho, \tilde{\pi})$) and Theorem 2 in [10],

$$\begin{aligned}\Gamma(1 + I_{A\alpha}) &= \Gamma(1 + I_{A\alpha} I) + \Gamma(1 + I(A\rho)) = \overline{I_{A\alpha} I} + \mathrm{Ker}[\omega_{A\alpha}: I(A\rho) \to \rho] \\ &= \mathrm{Ker}[\omega: \bar{I}_{A\alpha} \to \rho]. \quad \blacksquare\end{aligned}$$

This can be applied to give the following description of $K_1(A\pi)$:

PROPOSITION 2.3. *Let* π *be a p-group, and set* $I = I(A\pi)$. *Then*

$$K_1(A\pi) \cong A^* \oplus K_1(A\pi, I)$$

(*canonically*), *and there is a natural exact sequence*

$$0 \to K_1(A\pi, I) \xrightarrow{\mu} (A \otimes \pi^{ab}) \oplus Wh(A\pi) \xrightarrow{\nu} A \otimes \pi^{ab}.$$

Here, $\mu(1 + x) = (\omega^*_{A\pi}(x), [1 + x])$ for $x \in I$, *and*

$$\begin{aligned}\nu(\lambda \otimes g, [1 + x]) &= ((1 - \varphi) \otimes 1)(\lambda \otimes g - \omega^*_{A\pi}(x)) \\ &= (\lambda - \varphi(\lambda)) \otimes g - \omega^*_{A\pi} \Gamma_{A\pi}(1 + x).\end{aligned}$$

By $\omega^*_{A\pi}: I(A\pi) \to A \otimes \pi^{ab}$ *is meant* $\omega^*_{A\alpha}$ *when* $\alpha: \pi \to 1$.

Proof. We first check that the two formulas for ν agree, i.e., that

$$\omega^*_{A\pi} \Gamma_{A\pi}(1 + x) = ((1 - \varphi) \otimes 1)(\omega^*_{A\pi}(x)) \in A \otimes \pi^{ab}$$

for any $x \in I$. It clearly suffices to do this when π is abelian, and by Proposition 2.2 it suffices to consider the cases (a) $x \in I^2$ and (b) $x = \lambda(g-1)$ ($\lambda \in A$, $g \in \pi$). In case (a),

$$\omega^*_{A\pi}(x) = 0 \qquad \text{and} \qquad \Gamma_{A\pi}(1+x) \in I^2 = \mathrm{Ker}(\omega^*_{A\pi})$$

by Proposition 2.2 and Theorem 1.1. In case (b),

$$\begin{aligned}
&\Gamma(1+\lambda(g-1)) \\
&\quad = \frac{1}{p}\log\left(\frac{(1+\lambda(g-1))^p}{1+\varphi(\lambda)(g^p-1)}\right) \\
&\quad \equiv \frac{1}{p}\log\left(\frac{1+\lambda^p(g^p-1)+p(\lambda-\lambda^p)(g-1)}{1+\varphi(\lambda)(g^p-1)}\right) \qquad (\mathrm{mod}\ I_2) \\
&\quad \equiv \frac{1}{p}\log(1+(\lambda^p-\varphi(\lambda))(g^p-1)+p(\lambda-\lambda^p)(g-1)) \\
&\quad \equiv \frac{1}{p}(\lambda^p-\varphi(\lambda))(g^p-1)+(\lambda-\lambda^p)(g-1) \qquad (\mathrm{mod}\ I^2),
\end{aligned}$$

and so

$$\begin{aligned}
\omega^*_{A\pi}\Gamma_{A\pi}(1+\lambda(g-1)) &= \frac{1}{p}(\lambda^p-\varphi(\lambda))\otimes g^p+(\lambda-\lambda^p)\otimes g \\
&= (\lambda-\varphi(\lambda))\otimes g.
\end{aligned}$$

Since $\nu(Wh(A\pi)) \subseteq \nu(A\otimes\pi^{ab})$,

$$\mathrm{Ker}(\nu) = \langle (1\otimes g, 1), (\omega^*_{A\pi}(x), [1+x]) \colon g \in \pi, x \in I(A\pi)\rangle = \mathrm{Im}(\mu).$$

Furthermore,

$$\mathrm{Ker}(\mu) \subseteq \mathrm{Ker}[K_1(A\pi, I) \to Wh(A\pi)] = \pi^{ab},$$

but $\mu(g) = (1\otimes g, 1)$ for $g \in \pi$, and so μ is injective.

The isomorphism $K_1(A\pi) \cong K_1(A\pi, I) \oplus A^*$ is induced by the inclusion $A \subseteq A\pi$ and the augmentation map $A\pi \to A$. ∎

One more technical lemma will be needed:

LEMMA 2.4. *Let G be any group, H, $K \lhd G$ normal subgroups such that $H \cap K = 1$, and set*

$$I_H = \mathrm{Ker}[AG \to A[G/H]] \qquad \textit{and} \qquad I_K = \mathrm{Ker}[AG \to A[G/K]].$$

Then $I_H \cap I_K = I_H I_K$. Furthermore, if $S \subseteq G$ is a set of coset representatives for $H \times K \subseteq G$, then $I_H \cap I_K$ is a free A-module with basis

$$X = \{(h-1)(k-1)\,g: h \in H-1,\, k \in K-1,\, g \in S\}.$$

Proof. Clearly, $AX \subseteq I_H I_K \subseteq I_H \cap I_K$. Each element of $G-(HS \cup KS)$ (i.e., each element of the form hkg for $h \in H-1$, $k \in K-1$, $g \in S$) occurs as a term in exactly one element of X, and each element of X has exactly one such term. This implies first that the elements of X are A-linearly independent, and second that any $x \in AG$ is congruent $(\operatorname{mod} AX)$ to some $y = \sum \lambda_i g_i$ for $\lambda_i \in A$ and $g_i \in HS \cup KS$. In particular, if $x \in I_H \cap I_K$, then $y = 0$. In other words, $I_H \cap I_K \subseteq AX$, and we are done. ∎

THEOREM 2.5. *Let $1 \to \rho \to \tilde{\pi} \to^{\alpha} \pi \to 1$ be an extension of p-groups, set*

$$I_{A\alpha} = \operatorname{Ker}[A\tilde{\pi} \to A\pi] \qquad \text{and} \qquad M_{A\alpha} = \operatorname{Ker}[\omega_{A\alpha}: H_0(\rho; I_{A\alpha}) \to \rho^{ab}],$$

and let $\iota: M_{A\alpha} \to H_0(\rho; I_{A\alpha})$ be the inclusion. Then there is an exact sequence

$$SK_1(A\rho) \xrightarrow{s} K_1(A\tilde{\pi}, I_{A\alpha}) \xrightarrow{\Psi} A \otimes (\rho/[\tilde{\pi}, \rho]) \oplus H_0(\pi; M_{A\alpha})$$
$$\xrightarrow{(1-\varphi)\otimes 1 - (\omega^*_{A\alpha}\iota)/\pi} A \otimes (\rho/[\tilde{\pi}, \rho])$$

which is natural with respect to maps between extensions. Here,

(i) *s is the composite $SK_1(A\rho) \subseteq K_1(A\rho, I(A\rho)) \to K_1(A\tilde{\pi}, I_{A\alpha})$, and*

(ii) *$\Psi = (\Psi_1, \Psi_2)$, where $\Psi_1(1+x) = \omega^*_{A\alpha}(x)$.*

Furthermore, the composite

$$H_2(\pi) \xrightarrow{\lambda_\pi} K_2(A\pi)/\{-1, \pi\} \xrightarrow{\partial} K_1(A\tilde{\pi}, I_{A\alpha})$$
$$\xrightarrow{\Psi} A \otimes (\rho/[\tilde{\pi}, \rho]) \oplus H_0(\pi; M_{A\alpha})$$

(where λ_π denotes the map $\lambda''(\pi)$ defined in Section 4.3 of [8]) is equal to the pair $(\delta^\alpha, 0)$.

Proof. Define a group G and order $\mathfrak{A}$ as pullback:

$$\begin{array}{ccc} G & \xrightarrow{r_2} & \tilde{\pi} \\ \downarrow{\scriptstyle r_1} & & \downarrow{\scriptstyle \alpha} \\ \tilde{\pi} & \xrightarrow{\alpha} & \pi \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \mathfrak{A} & \xrightarrow{r_2} & A\tilde{\pi} \\ \downarrow{\scriptstyle r_1} & & \downarrow \\ A\tilde{\pi} & \longrightarrow & A\pi \end{array}$$

Let $\Delta\colon \tilde{\pi} \to G$ be the diagonal map (regarding G as a subgroup of $\tilde{\pi} \times \tilde{\pi}$). Set $\rho_i = \mathrm{Ker}(r_i) \subseteq G$, and note that $\rho_1 \cong \rho_2 \cong \rho$ and $G/(\rho_1 \times \rho_2) \cong \pi$. Set

$$I_1 = \mathrm{Ker}[r_1\colon AG \to A\tilde{\pi}] \qquad \text{and} \qquad I_2 = \mathrm{Ker}[r_2\colon AG \to A\tilde{\pi}].$$

Then $\mathfrak{A} \cong AG/(I_1 \cap I_2)$ (since $\mathfrak{A} \subseteq A\tilde{\pi} \times A\tilde{\pi}$).

Step 1. By Proposition 2.3, there is an exact sequence

$$0 \to K_1(AG) \to A^* \oplus (A \otimes G^{ab}) \oplus Wh(AG) \xrightarrow[-\omega_{AG}^* \Gamma_{AG}]{(1-\varphi)\otimes 1} A \otimes G^{ab}. \qquad (1)$$

Furthermore, $K_1(\mathfrak{A}) \cong K_1(AG)/(1 + I_1 I_2)$ ($I_1 \cap I_2 = I_1 I_2$ by Lemma 2.4), and

$$I_1 I_2 \subseteq I(AG)^2 = \mathrm{Ker}(\omega_{AG}^*)$$

by Lemma 2.2. So upon dividing (1) out by $1 + I_1 I_2$, we get the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
0 & \to & K_1(\mathfrak{A}) & \to & A^* \oplus (A \otimes G^{ab}) \oplus \dfrac{Wh(AG)}{1 + I_1 I_2} & \to & A \otimes G^{ab} \\
 & & \Big\downarrow {\scriptstyle K_1(r_1)} & & \Big\downarrow {\scriptstyle (1,1 \otimes r_1^{ab}, Wh(r_1))} & & \Big\downarrow {\scriptstyle 1 \otimes r_1^{ab}} \\
0 & \to & K_1(A\tilde{\pi}) & \to & A^* \oplus (A \otimes \tilde{\pi}^{ab}) \oplus Wh(A\tilde{\pi}) & \to & A \otimes \tilde{\pi}^{ab}
\end{array} \qquad (2)$$

For convenience, we set $Wh(\mathfrak{A}) = Wh(AG)/(1 + I_1 I_2)$.

Now, $K_1(A\tilde{\pi}, I_{A\alpha}) = \mathrm{Ker}(K_1(r_1))$ by definition [9]. Furthermore, since r_1 is split by Δ,

$$\mathrm{Ker}(r_1^{ab}) = \rho_1/[G, \rho_1] = H_0(G; \rho_1^{ab}) \cong H_0(\pi; \rho^{ab}) = \rho/[\tilde{\pi}, \rho].$$

So the kernels in (2) form an exact sequence

$$0 \to K_1(A\tilde{\pi}, I_{A\alpha}) \xrightarrow{\Psi'} A \otimes (\rho/[\tilde{\pi}, \rho]) \oplus \mathrm{Ker}(Wh_1(r_1))$$

$$\xrightarrow[-\omega_{AG}^* \Gamma_{AG}]{(1-\varphi)\otimes 1} A \otimes \rho/[\tilde{\pi}, \rho]. \qquad (3)$$

Step 2. Now let $1 \to \sigma \to \hat{G} \to^{\beta} G \to 1$ be any central extension such that

(a) $\mathrm{Ker}(\delta^\beta) = \mathrm{Im}[H_2(\rho_1 \times \rho_2) \oplus H_2(\tilde{\pi}) \xrightarrow{\mathrm{incl} + \Delta^*} H_2(G)]$, and

(b) Δ lifts to a map $\hat{\Delta}\colon \tilde{\pi} \to \hat{G}$.

For example, $(\hat{G}, \beta)$ can be constructed by first choosing any central extension

$$1 \to \sigma \to G' \xrightarrow{\beta'} G \to 1$$

satisfying (a) (see Proposition V.5.1 in [12]), and then setting

$$[\hat{G}, \beta] = [G', \beta'] - r_2^* \Delta^* [G', \beta'] \in H^2(G, \sigma).$$

Then $\delta^\beta = \delta^{\beta'}$ ($\delta^{\beta'} \circ \Delta_* = 0$ by (a)), and $\Delta^*[\hat{G}, \beta] = 0$.

Set $\hat{\rho}_i = \beta^{-1}(\rho_i)$ $(i = 1, 2)$, and note that $[\hat{\rho}_1, \hat{\rho}_2] = 1$ by construction. Furthermore, by (a) and Proposition 2.1(iv),

$$\mathrm{incl} + \Delta_*: SK_1(A\hat{\rho}_1) \oplus SK_1(A\tilde{\pi}) \to SK_1(A\hat{G}) \tag{4}$$

is onto, and $SK_1(A\hat{\rho}_1) \cong SK_1(A\rho_1)$. Define ideals

$$J = I(A\hat{G}), \qquad J_0 = \mathrm{Ker}[\beta: A\hat{G} \to AG],$$
$$J_i = \mathrm{Ker}[r_i\beta: A\hat{G} \to A\tilde{\pi}] \qquad (i = 1, 2).$$

Then $AG = A\hat{G}/J_0$ and $I_i = J_i/J_0$ $(i = 1, 2)$, and so

$$Wh(\mathfrak{A}) = Wh(AG)/(1 + I_1 I_2) \cong Wh(A\hat{G})/(1 + J_0 + J_1 J_2).$$

By Theorem 1.1 and Proposition 2.2(ii),

$$\Gamma(1 + J_0 + J_1 J_2) = \mathrm{Ker}[\omega_{A\beta}: J_0 \to \sigma] + J_1 J_2$$

(or rather, its image in $\overline{I(A\hat{G})}$).

Now, identify $I(A\hat{G})$ with $H_0(G; I(A\hat{G}))$ (G acting by conjugation) and write $K = \mathrm{Ker}(\omega_{A\beta})$ for short. Then Theorem 2 in [10] applies to give the following diagram with exact rows:

$$\begin{array}{ccccccccccc}
 & & SK_1(A\hat{G}) & \longrightarrow & Wh(\mathfrak{A}) & \xrightarrow{\Gamma_{A\hat{G}}} & H_0(G; J/(K + J_1 J_2)) & \xrightarrow{\omega_{A\hat{G}}} & \hat{G}^{ab} & \longrightarrow \\
 & & \downarrow{\scriptstyle SK_1(r_1\beta)} & & \downarrow{\scriptstyle Wh(r_1)} & & \downarrow{\scriptstyle \overline{I(r_1)}} & & \downarrow{\scriptstyle r_1^{ab}} & \\
0 & \longrightarrow & SK_1(A\tilde{\pi}) & \longrightarrow & Wh(A\tilde{\pi}) & \xrightarrow{\Gamma_{A\tilde{\pi}}} & H_0(G; I(A\tilde{\pi})) & \xrightarrow{\omega_{A\tilde{\pi}}} & \tilde{\pi}^{ab} & \longrightarrow
\end{array}$$

As in (2), the vertical maps are all split by maps induced by Δ or $\hat{\Delta}$, and so the kernels form an exact sequence. By (4), $\mathrm{Ker}(SK_1(r_1\beta))$ is generated by $SK_1(A\hat{\rho}_1) \cong SK_1(A\rho_1)$. It follows that $\Gamma_{A\hat{G}}$ induces an isomorphism (where $\hat{r}_1 = r_1\beta: \hat{G} \to \tilde{\pi}$)

$$\mathrm{Ker}\left[\frac{Wh(\mathfrak{A})}{SK_1(A\rho_1)} \xrightarrow{r_{1*}} Wh(A\tilde{\pi})\right]$$
$$\cong \mathrm{Ker}\left[\omega_{A\hat{r}_1}: H_0\left(G; \frac{J_1}{K + J_1 J_2}\right) \to \hat{\rho}_1/[\hat{G}, \hat{\rho}_1]\right]. \tag{5}$$

Step 3. Now consider the group

$$\operatorname{Ker}\left[r_2\colon \frac{J_1}{K+J_1J_2} \to I_{A\alpha}\right] = \frac{J_1 \cap J_2}{K+J_1J_2} = \frac{J_0+J_1J_2}{K+J_1J_2}$$
$$\cong \frac{J_0}{K+(J_0 \cap J_1J_2)}.$$

Using Lemma 2.4 (the fact that X is a basis for $I_1I_2 = J_1J_2/J_0$), we see that $J_0 \cap J_1J_2$ is generated by elements

$$\begin{aligned}(1-h)(1-k)\,g &- (1-h')(1-k')\,g \\ &= (h'-h)(1-k)\,g + (1-h')(k'-k)\,g \in JJ_0\end{aligned}$$

(where $h \in \hat{\rho}_1$, $k \in \hat{\rho}_2$, $g \in \hat{G}$, and $h^{-1}h'$, $k^{-1}k' \in \sigma$). Since $JJ_0 \subseteq K$,

$$\operatorname{Ker}[r_2\colon J_1/(K+J_1J_2) \to I_{A\alpha}] \cong J_0/K = J_0/\operatorname{Ker}(\omega_{A\beta}) \cong \sigma.$$

In other words, there is a short exact sequence

$$0 \to \sigma \xrightarrow{f} J_1/(K+J_1J_2) \xrightarrow{r_2} I_{A\alpha} \to 0, \tag{6}$$

where $f(g) = \lambda(g-1)$ for any $g \in \sigma$, and any $\lambda \in A$ such that $\operatorname{Tr}(\lambda) = 1$.

Step 4. Write $\hat{\rho} = \hat{\rho}_1 = \operatorname{Ker}[\hat{r}_1\colon G \to \tilde{\pi}]$,. and consider the maps

$$\omega_{A\alpha}\colon I_{A\alpha} \to \rho^{ab} \qquad \text{and} \qquad \omega_{A\hat{r}_1}\colon J_1 \to \hat{\rho}^{ab}$$

($J_1 = \operatorname{Ker}[\hat{r}_1\colon A\hat{G} \to A\tilde{\pi}]$ defined as before). Since

$$K + J_1J_2 = \operatorname{Ker}(\omega_{A\beta}) + J_1J_2 \subseteq \operatorname{Ker}(\hat{\omega}),$$

there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & \sigma & \xrightarrow{f} & H_0(\rho_1\times\rho_2; J_1/(K+J_1J_2)) & \xrightarrow{r_2} & H_0(\rho; I_{A\alpha}) & \to & 0 \\ & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \hat{\omega}} & & \downarrow{\scriptstyle \omega} & & \\ 0 & \to & \sigma & \longrightarrow & \hat{\rho}^{ab} & \xrightarrow{r_2} & \rho^{ab} & \to & 0 \end{array} \tag{7}$$

Both rows are right exact: the top row by (6) and the bottom row by the homology sequence for the extension

$$1 \to \sigma \to \hat{\rho} \xrightarrow{\beta_2} \rho \to 1$$

where $\beta_2 = (r_2\beta) \mid \hat{\rho}$. But $\delta^{\beta_2} = 0$ by construction (see (a)), and so the rows in (7) are also exact at σ.

It follows that

$$\mathrm{Ker}(\hat{\omega}) \cong \mathrm{Ker}(\omega) = M_{A\alpha}.$$

Hence, there is an exact sequence

$$\mathrm{Coker}(H_1(\pi; \hat{\omega})) \to H_0(\pi; M_{A\alpha}) \to H_0(G; J_1/(K + J_1J_2)) \to H_0(\pi; \hat{\rho}^{ab}). \quad (8)$$

The homology spectral sequences for $\tilde{\pi} \to^{\alpha} \pi$ and $G \to^{\bar{\alpha}} \pi$ induce a commutative diagram

$$\begin{tikzcd}
H_3(\pi) \arrow[r, "\partial"] \arrow[d, "\mathrm{id}"] & H_1(\pi; \rho^{ab}) \arrow[r] \arrow[d, "\mathrm{diag}"] & H_2(\tilde{\pi})/H_2(\rho) \arrow[r] \arrow[d, "\Delta_*"] & H_2(\pi) \arrow[d, "\mathrm{id}"] \\
H_3(\pi) \arrow[r, "\partial"] & H_1(\pi; \rho_1^{ab} \times \rho_2^{ab}) \arrow[r] & H_2(G)/H_2(\rho_1 \times \rho_2) \arrow[r] & H_2(\pi)
\end{tikzcd}$$

where the vertical maps are split injective (split by r_2). So

$$H_1(\pi; \rho^{ab}) \cong \mathrm{Coker}[\mathrm{incl} + \Delta_*: H_2(\rho_1 \times \rho_2) \oplus H_2(\tilde{\pi}) \to H_2(G)],$$

and by (a) in Step 2,

$$H_1(\pi; \rho^{ab}) \cong \mathrm{Im}[\delta^{\beta}: H_2(G) \to \sigma]. \quad (9)$$

On the other hand, if $\partial: H_1(\pi; \rho^{ab}) \to \sigma$ is the boundary map induced by the π-action on the bottom row of (7), then

$$\mathrm{Im}(\partial) = \mathrm{Ker}[\sigma \to H_0(\pi; \hat{\rho}^{ab})] = \mathrm{Ker}[\sigma \to \hat{G}^{ab}] = \mathrm{Im}(\delta^{\beta}): \quad (10)$$

$H_0(\pi; \hat{\rho}^{ab})$ injects into $\hat{G}^{ab}$ since $\hat{G} \to \tilde{\pi}$ is split by $\hat{\Delta}$. So by (9) and (10), ∂ is injective.

Referring again to diagram (7), this says that

$$\mathrm{Coker}[H_1(\pi; \sigma) \to H_1(\pi; \hat{\rho}^{ab})] \cong \mathrm{Ker}[\partial: H_1(\pi; \rho^{ab}) \to \sigma] = 0.$$

So $H_1(\pi; \hat{\omega})$ is onto, and (8) takes the form

$$H_0(\pi; M_{A\alpha}) \cong \mathrm{Ker}[\hat{\omega}: H_0(G; J_1/(K + J_1J_2)) \to H_0(\pi; \hat{\rho}_1^{ab})]. \quad (11)$$

Combining this with (5) and (3) gives, finally, the exact sequence

$$\begin{aligned}
0 \to K_1(A\tilde{\pi}, I_{A\alpha})/SK_1(A\rho) &\xrightarrow{\Psi} A \otimes (\rho/[\tilde{\pi}, \rho)] \oplus H_0(\pi; M_{A\alpha}) \\
&\to A \otimes (\rho/[\tilde{\pi}, \rho]). \quad (12)
\end{aligned}$$

The descriptions of the maps in (12) follow directly from the constructions. That Ψ is uniquely defined (independently of the choice of $\hat{G}$) and natural follows by taking pullbacks: for example, if $\hat{G}_1$ and $\hat{G}_2$ are two central extensions of G satisfying conditions (a) and (b) in Step 2, then their pullback over G also satisfies (a) and (b).

Finally, the description of $\Psi\,\partial\lambda_\pi$ follows from the construction of (3) and the commutativity of the square:

$$\begin{tikzcd} H_2(\pi) \arrow[r, "\delta^\alpha"] \arrow[d, "\lambda_\pi"] & \rho/[\tilde{\pi}, \rho] \arrow[d] \\ K_2(A\pi)/\{-1, \pi\} \arrow[r, "\partial"] & K_1(A\tilde{\pi}, I_{A\alpha}) \end{tikzcd} \quad \blacksquare$$

A description of the map $\Psi_2\colon K_1(A\tilde{\pi}, I_{A\alpha}) \to H_0(\pi; M_{A\alpha})$ in Theorem 2.5 can be extracted from the proof of the theorem, but is technically quite difficult to write down. The composite

$$(\iota/\pi)\circ\Psi_2\colon K_1(A\tilde{\pi}, I_{A\alpha}) \to H_0(\tilde{\pi}; I_{A\alpha})$$

is, on the other hand, much more natural, and is basically a relative version of the modified logarithm map $\Gamma_{A\tilde{\pi}}$. More precisely:

PROPOSITION 2.6. *Let $1 \to \rho \to \tilde{\pi} \to^\alpha \pi \to 1$ be any extension of p-groups, let $G \subseteq \tilde{\pi}\times\tilde{\pi}$ and $\mathfrak{A} \subseteq A\tilde{\pi}\times A\tilde{\pi}$ be the pullbacks over π and $A\pi$, and let r_1 and r_2 be the projections of $\mathfrak{A}$ onto $A\tilde{\pi}$. Let $\gamma\colon AG \to \mathfrak{A}$ be the projection, and define*

$$\Gamma_{A\alpha}\colon K_1(A\tilde{\pi}, I_{A\alpha}) \to H_0(\tilde{\pi}; I_{A\alpha}) \qquad (I_{A\alpha} = \mathrm{Ker}[A\tilde{\pi} \to A\pi])$$

to be the composite

$$\begin{aligned} K_1(A\tilde{\pi}, I_{A\alpha}) &\xrightarrow{r_{2*}^{-1}} \mathrm{Ker}[r_{1*}\colon K_1(\mathfrak{A}) \to K_1(A\tilde{\pi})] \\ &\twoheadleftarrow \mathrm{Ker}[r_1\gamma\colon K_1(AG) \to K_1(A\tilde{\pi})] \\ &\xrightarrow{\Gamma_{AG}} \mathrm{Ker}[r_1\gamma\colon \overline{I(AG)} \to \overline{I(A\tilde{\pi})}] \cong H_0(G; I_{Ar_1}) \\ &\xrightarrow{r_2} H_0(\tilde{\pi}; I_{A\alpha}). \end{aligned}$$

Then $\Gamma_{A\alpha}$ is well defined, and $\Gamma_{A\alpha} = (\iota/\pi)\,\Psi_2$ in the notation of Theorem 2.5.

Proof. That $(\iota/\pi)\,\Psi_2$ equals the above composite follows from the construction of Ψ_2 in the proof of Theorem 2.5. That $\Gamma_{A\alpha}$ is well defined thus follows automatically (but can also be shown directly, using Theorem 1.1). $\blacksquare$

3

We still let A be a fixed unramified p-ring, with $\varphi \in \mathrm{Gal}(A/\hat{\mathbb{Z}}_p)$ and $\mathrm{Tr} = \mathrm{Tr}_{A/\mathbb{Z}_p}$ as in Section 2. Theorem 2.5 will now be applied to calculate

$$K_2^*(A\pi) \cong \varprojlim \mathrm{Ker}[K_1(A\tilde{\pi}, I_{A\alpha}) \to K_1(A\tilde{\pi})],$$

where π is a p-group and the limit is taken over all extensions

$$1 \to \rho \to \tilde{\pi} \xrightarrow{\alpha} \pi \to 1$$

of π by p-groups. For any such extension (not necessarily of p-groups), we set

$$I_{A\alpha} = \mathrm{Ker}[A\tilde{\pi} \to A\pi] \qquad \text{and} \qquad M_{A\alpha} = \mathrm{Ker}[\omega_{A\alpha}\colon H_0(\rho; I_{A\alpha}) \to \hat{\mathbb{Z}}_p \otimes \rho^{ab}]$$

(where $\omega_{A\alpha}(\lambda(h-1)g) = \mathrm{Tr}(\lambda) \otimes h$ for $\lambda \in A$, $h \in \rho$, and $g \in \tilde{\pi}$). Let

$$\iota_{A\alpha}\colon M_{A\alpha} \to H_0(\rho; I_{A\alpha}) \qquad \text{and} \qquad \kappa_{A\alpha}\colon H_0(\pi; M_{A\alpha}) \to \overline{I(A\tilde{\pi})}$$

be the maps induced by the inclusions. The main problem now is to study $\mathrm{Ker}(\kappa_{A\alpha})$, and its limit over all such $(\tilde{\pi}, \alpha)$.

For simplicity, the subscripts in $\kappa_{A\alpha}$, $\omega_{A\alpha}$, etc., will be dropped whenever possible.

LEMMA 3.1. *For any extension $1 \to \rho \to \tilde{\pi} \to^{\alpha} \pi \to 1$ of groups there is an exact sequence*

$$H_1(\tilde{\pi}; I_{A\alpha}) \xrightarrow{\Omega} H_1(\pi; \hat{\mathbb{Z}}_p \otimes \rho^{ab}) \to \mathrm{Ker}[\kappa\colon H_0(\pi; M_{A\alpha}) \to \overline{I(A\tilde{\pi})}]$$

$$\xrightarrow{\iota/\pi} \mathrm{Ker}[H_0(\pi; I_{A\alpha}) \to \overline{I(A\tilde{\pi})}]$$

$$\xrightarrow{\omega} \hat{\mathbb{Z}}_p \otimes \mathrm{Ker}[\rho/[\tilde{\pi}, \rho] \to \pi^{ab}],$$

where $\Omega = \Omega_{A\alpha}$ is induced by $\omega = \omega_{A\alpha}$.

Proof. This follows directly from the homology sequence of the short exact sequence of π-modules

$$0 \to M_{A\alpha} \xrightarrow{\iota} H_0(\rho; I_{A\alpha}) \xrightarrow{\omega} \hat{\mathbb{Z}}_p \otimes \rho^{ab} \to 0.$$

Note that $H_1(\tilde{\pi}; I_{A\alpha}) \to H_1(\pi; H_0(\rho; I_{A\alpha}))$ is onto (look at the spectral sequence). ∎

The next lemma is standard, but will be needed frequently for handling homology with coefficients in parmutation modules.

LEMMA 3.2. *Let G be a group, and S a set upon which G acts. Let $A(S)$ denote the free A-module with basis S, regarded as an AG-module. Let $X \subseteq S$ be a set of G-orbit representatives, and define, for all $x \in X$,*

$$G_x = \{g \in G\colon g(x) = x\}.$$

Then, for all n,

$$H_n(G; A(S)) \cong \sum_{x \in X} H_n(G_x; A).$$

In particular, $H_1(G; A(S))$ is generated by elements of the form $g \otimes x$ for $x \in S$ and $g \in G_x$.

Proof. Just note that

$$A(S) \cong \sum_{x \in X} \operatorname{Ind}_{G_x}^G(A),$$

so the result follows by Shapiro's lemma (see, e.g., [16, p. 131]). ∎

Now let π be a p-group again. For any unramified p-ring A, we define a group $\mathcal{U}(A\pi)$ as in the Introduction:

$$\mathcal{U}(A\pi) = H_1(\pi; A\pi)/\langle g \otimes \lambda g^n\colon g \in \pi,\ \lambda \in A,\ n \in \mathbb{Z}\rangle,$$

where π acts on $A\pi$ via conjugation. By Lemma 3.2,

$$\mathcal{U}(A\pi) \cong \sum_i (Z_\pi(g_i)/S_\pi(g_i))^{ab} \otimes A(g_i),$$

where g_i runs over a set of conjugacy class representatives for π and

$$S_\pi(g) = \langle \sigma \subseteq \pi\colon \sigma \text{ cyclic},\ g \in \sigma\rangle = \langle h\colon g = h^n, \text{ some } n\rangle.$$

We write $\mathcal{U}(\pi) = \mathcal{U}(\mathbb{Z}_p\pi)$ for short. Note that $\mathcal{U}(A\pi) \cong A \otimes \mathcal{U}(\pi)$ in general.

PROPOSITION 3.3. *Let $1 \to \rho \to \tilde{\pi} \xrightarrow{\alpha} \pi \to 1$ be any extension of p-groups such that $\alpha(\sigma)$ is cyclic for any abelian $\sigma \subseteq \pi$. Define*

$$\theta_0\colon \mathcal{U}(A\pi) \to H_0(\tilde{\pi}; I_{A\alpha})$$

by setting, for any $\lambda \in A$, commuting $g, h \in \pi$, and $\tilde{g} \in \alpha^{-1}g$, $\tilde{h} \in \alpha^{-1}h$,

$$\theta_0(h \otimes \lambda g) = \lambda(\tilde{h}\tilde{g}\tilde{h}^{-1} - \tilde{g}) \in I_{A\alpha} = \operatorname{Ker}[A\tilde{\pi} \to A\pi].$$

Then θ_0 is well defined, and induces an isomorphism

$$\theta\colon \mathcal{U}(A\pi) \xrightarrow{\cong} \operatorname{Ker}[H_0(\tilde{\pi}; I_{A\alpha}) \to \overline{I(A\tilde{\pi})}].$$

Proof. Let $\tilde{\theta}$ denote the composite

$$\tilde{\theta}\colon H_1(\tilde{\pi}; A\pi) \twoheadrightarrow H_1(\pi; A(\pi)) \twoheadrightarrow \mathcal{U}(A\pi).$$

Consider the diagram

$$\begin{tikzcd} H_1(\tilde{\pi}; A\tilde{\pi}) \arrow[r, "\alpha_*"] & H_1(\tilde{\pi}; A\pi) \arrow[r, "\partial"] \arrow[dr, "\tilde{\theta}"'] & H_0(\tilde{\pi}; I_{A\alpha}) \arrow[r, "i_*"] & H_0(\tilde{\pi}; A\tilde{\pi}) \supseteq \overline{I(A\tilde{\pi})} \\ & & \mathcal{U}(A\pi) \arrow[u, dashed, "\theta_0"'] & \end{tikzcd}$$

where the row is induced by the short exact sequence

$$0 \to I_{A\alpha} \to A\tilde{\pi} \to A\pi \to 0$$

of $\tilde{\pi}$-modules. We defined θ_0 by the relation $\theta_0(\tilde{\theta}(x)) = \partial(x)$. To see that this is well defined and induces an isomorphism $\mathcal{U}(A\pi) \cong \operatorname{Ker}(i_*)$, it will suffice to show that $\operatorname{Ker}(\tilde{\theta}) = \operatorname{Im}(\alpha_*)$.

By Lemma 3.2,

$$\operatorname{Im}(\alpha_*) = \langle h \otimes \lambda\alpha(g) \in H_1(\tilde{\pi}; A\pi)\colon \lambda \in A,\ h, g \in \tilde{\pi},\ hg = gh \rangle.$$

On the other hand

$$\operatorname{Ker}(\tilde{\theta}) = \langle h \otimes \lambda g \in H_1(\tilde{\pi}; A\pi)\colon \lambda \in A,\ h \in \tilde{\pi},\ g \in \pi, \text{ and } \alpha(h) \in S_\pi(g) \rangle.$$

So we must show that the subgroups

$$H_g = \langle h \in \tilde{\pi}\colon g \in \alpha Z_{\tilde{\pi}}(h) \rangle \subseteq \tilde{\pi} \qquad \text{and} \qquad \alpha^{-1} S_\pi(g) \subseteq \tilde{\pi}$$

are equal for all $g \in \pi$.

Clearly, $\alpha^{-1} g \subseteq H_g$, and so $\rho \subseteq \langle \alpha^{-1} g \rangle \subseteq H_g$. It thus suffices to show that $\alpha(H_g) = S_\pi(g)$. But

$$\begin{aligned} \alpha(H_g) &= \langle h \in \pi\colon \tilde{h}\tilde{g} = \tilde{g}\tilde{h} \text{ for some } \tilde{h} \in \alpha^{-1}h,\ \tilde{g} \in \alpha^{-1}g \rangle \\ &= \langle h \in \pi\colon g, h \text{ generate a cyclic subgroup} \rangle \\ &= \langle h \in \pi\colon h^n = g, \text{ some } n \rangle = S_\pi(g). \quad \blacksquare \end{aligned}$$

In order to compute limits, we will need to know the existence of certain "sufficiently large" extensions of a given group; in particular, the existence of extensions meeting the hypotheses of Proposition 3.3. This is done in the following lemma:

LEMMA 3.4. *For any p-group π, there is an extension*

$$1 \to \rho \to \tilde{\pi} \xrightarrow{\alpha} \pi \to 1 \tag{1}$$

of p-groups having the properties

(i) *For any abelian $\sigma \subseteq \tilde{\pi}$, $\alpha(\sigma)$ is cyclic.*

(ii) *For all $n \geqslant 2$, the boundary map*

$$\partial_n\colon H_n(\pi) \to H_{n-2}(\tilde{\pi}; \rho^{ab})$$

(in the spectral sequence for (1)) *is injective (note that $\partial_2 = \delta^\alpha$).*

In fact, if $\pi = F/R$ where F is free, then $\tilde{\pi}$ can be chosen of the form F/S for some $S \subseteq R$.

Proof. Write $\pi = F/R$ where F is free and finitely generated. Let $q = |\pi|$, and set

$$S = \langle [R, R], R^q \rangle \qquad \text{and} \qquad \tilde{\pi} = F/S.$$

Then $R/S \cong R^{ab} \otimes \mathbb{Z}/q$, so $\tilde{\pi}$ is a p-group. Let $\alpha\colon \tilde{\pi} \to \pi$ be the projection.

(i) We claim that $\alpha(\sigma)$ is cyclic for any abelian $\sigma \subseteq \tilde{\pi}$. To see this, choose $g, h \in F$, set $F_0 = \langle R, g, h \rangle$, and assume that $F_0/R \subseteq \pi$ is not cyclic. We must show that $[g, h] \notin S$.

Since F_0/R is a non-cyclic p-group, we may choose $R_0 \lhd F_0$ such that $R_0 \supseteq R$, $F_0/R_0 \cong \mathbb{Z}/p \times \mathbb{Z}/p$, and F_0/R_0 is generated by g and h. Then $g \wedge h$ is non-zero in $H_2(F_0/R_0) \cong \mathbb{Z}/p$, and the composite

$$H_2(F_0/R_0) \xrightarrow{\delta} \frac{R_0}{[F_0, R_0]} \to \frac{R_0}{[F_0, R_0]} \otimes \mathbb{Z}/p$$
$$= R_0/\langle [F_0, R_0], R_0^p \rangle$$

is injective by Proposition 2.1(ii). Since $\delta(g \wedge h) = [g, h]$ (by definition of $g \wedge h$), this implies that

$$[g, h] \notin \langle [F_0, R_0], R_0^p \rangle \supseteq \langle [R, R], R^q \rangle = S.$$

(ii) Recall that $\exp(H_2(\pi))$ divides $|\pi| = q$ (see Proposition XII.2.5 in [3]). Hence

$$\delta^\alpha\colon H_2(\pi) \to H_0(\pi; (R/S)^{ab}) \cong (R/[R, F]) \otimes \mathbb{Z}/q$$

is injective by Proposition 2.1(ii). For $n \geqq 3$,

$$\partial\colon H_n(\pi) \to H_{n-2}(\pi; R^{ab})$$

is an isomorphism since $H_i(F) \cong H_i(R) = 0$ for $i \geqq 2$. But

$$\exp H_{n-2}(\pi; R^{ab}) \mid q$$

(see [3] again), so $H_{n-2}(\pi; R^{ab})$ injects into $H_{n-2}(\pi; R^{ab} \otimes \mathbb{Z}/q)$, and we are done. ∎

Lemma 3.4 can now be applied to describe some of the limits which arise in computing $K_2^*(A\pi)$.

PROPOSITION 3.5. *Let $1 \to R \to F \xrightarrow{\alpha} \pi \to 1$ be an extension, where π is a p-group and F is free. Let $\mathscr{S}$ be the set of all subgroups $S \lhd F$ such that $S \subseteqq R$ and F/S is a p-group. For $S \in \mathscr{S}$, write $\tilde{\pi}_S = F/S$, $\rho_S = R/S$, and let $\alpha_S \colon \tilde{\pi}_S \to \pi$ be the projection. Then*

(i) $\varprojlim_{S \in \mathscr{S}} SK_1(A\tilde{\pi}_S) \cong \varprojlim_{S \in \mathscr{S}} SK_1(A\rho_S) = 0$,

(ii) $\varprojlim_{S \in \mathscr{S}} H_1(\pi; \rho_S^{ab}) \cong H_1(\pi; R^{ab}) \cong H_3(\pi)$, *and*

(iii) $\varprojlim_{S \in \mathscr{S}} \operatorname{Ker}[\kappa_{A\alpha_S} \colon H_0(\pi; M_{A\alpha_S}) \to \overline{I(A\tilde{\pi}_S)}]$

$\cong \operatorname{Ker}[\kappa_{A\alpha} \colon H_0(\pi; M_{A\alpha}) \to \overline{I(AF)}]$.

Proof. (i) Fix some $S \in \mathscr{S}$, set $q = |\tilde{\pi}_S|$, and define

$$S' = \langle [F, S], S^q \rangle \qquad \text{and} \qquad S'' = \langle [R, S], S^q \rangle.$$

Then $S', S'' \in \mathscr{S}$, since

$$S/S' \cong S/[F, S] \otimes \mathbb{Z}/q \qquad \text{and} \qquad S/S'' \cong S/[R, S] \otimes \mathbb{Z}/q$$

are p-groups. The boundary maps for the central extensions

$$1 \to S/S' \to \tilde{\pi}_{S'} \to \tilde{\pi}_S \to 1 \qquad \text{and} \qquad 1 \to S/S'' \to \rho_{S''} \to \rho_S \to 1$$

are injective by Proposition 2.1(ii), so $SK_1(A\tilde{\pi}_{S'}) \cong SK_1(A\rho_{S''}) = 0$ by Proposition 2.1(iv), and the limits vanish.

(ii) Since $H_n(F) \cong H_n(R) = 0$ for $n \geqslant 2$, the boundary map

$$\partial \colon H_3(\pi) \to H_1(\pi; R^{ab})$$

in the spectral sequence is an isomorphism. Furthermore, by Lemma 3.4,

$$\delta \colon H_2(\pi) \to H_0(\pi; \rho_S^{ab}) \qquad \text{and} \qquad \partial \colon H_3(\pi) \to H_1(\pi; \rho_S^{ab})$$

are injective for sufficiently small $S \in \mathscr{S}$. For such S, the spectral sequence for $\rho_S \lhd \tilde{\pi}_S$ induces a surjection

$$H_2(\tilde{\pi}_S) \twoheadrightarrow \operatorname{Coker}[\partial \colon H_3(\pi) \to H_1(\pi; \rho_S^{ab}),$$

and hence a surjection (all groups involved being finite)

$$0 = \varprojlim_{S \in \mathcal{S}} H_2(\tilde{\pi}_S) \twoheadrightarrow \operatorname{Coker}[\varprojlim \partial: H_3(\pi) \to \varprojlim_{S \in \mathcal{S}} H_1(\pi; \rho_S^{ab})].$$

So $\varprojlim(\partial)$ is an isomorphism.

(iii) Fix some $S_0 \in \mathcal{S}$ such that $\alpha_{S_0}: \tilde{\pi}_{S_0} \twoheadrightarrow \pi$ fulfills the hypotheses of Lemma 3.4. Then, for any $S \subseteq S_0$ in $\mathcal{S}$,

$$\operatorname{Ker}[H_0(\tilde{\pi}_S; I_{A\alpha_S}) \to \overline{I(A\tilde{\pi}_S)}] \cong \operatorname{Ker}[H_0(F; I_{A\alpha}) \to \overline{I(AF)}] \cong \mathcal{U}(A\pi)$$

by Proposition 3.3. Lemma 3.1 thus applies to give the following diagram with exact rows:

$$\begin{tikzcd}
H_1(F; I_{A\alpha}) \arrow[r] \arrow[d, "f_S"] & H_1(\pi; R^{ab}) \arrow[r] \arrow[d] & \operatorname{Ker}(\kappa_{A\alpha}) \arrow[r] \arrow[d, "f_S'"] & \mathcal{U}(A\pi) \arrow[r, "(\delta^\alpha)^{-1}\omega"] \arrow[d, "\mathrm{id}"] & H_2(\pi) \arrow[d, "\mathrm{id}"] \\
H_1(F; I_{A\alpha_S}) \arrow[r] & H_1(\pi; \rho_S^{ab}) \arrow[r] & \operatorname{Ker}(\kappa_{A\alpha_S}) \arrow[r] & \mathcal{U}(A\pi) \arrow[r] & H_2(\pi)
\end{tikzcd}$$

Together with (ii), this implies that

$$f': \operatorname{Ker}(\kappa_{A\alpha}) \to \varprojlim_{S \in \mathcal{S}} \operatorname{Ker}(\kappa_{A\alpha_S})$$

is surjective. Furthermore, the diagram allows us to identify $\operatorname{Ker}(f_S')$ with a subquotient of $\operatorname{Coker}(f_S)$, so f' is an isomorphism if we can show that for every $S \subseteq S_0$ in $\mathcal{S}$, there is $S' \subseteq S$ in $\mathcal{S}$ such that the map $\operatorname{Coker}(f_{S'}) \to \operatorname{Coker}(f_S)$ is zero.

Fix S, and choose $S' \subseteq S$ such that any abelian $\sigma \subseteq \tilde{\pi}_S$ has cyclic image in $\tilde{\pi}_S$. In particular, the induced map $\mathcal{U}(\tilde{\pi}_{S'}) \to \mathcal{U}(\tilde{\pi}_S)$ is zero. The extensions

$$0 \to I_1 \to I_{A\alpha} \to I_{A\alpha_{S'}} \to 0 \qquad \text{and} \qquad 0 \to I_2 \to I_{A\alpha} \to I_{A\alpha_S} \to 0$$

(where $I_1 = \operatorname{Ker}[AF \to A\tilde{\pi}_{S'}]$, $I_2 = \operatorname{Ker}[AF \to A\tilde{\pi}_S]$) induce inclusions

$$\begin{aligned}\operatorname{Coker}(f_{S'}) &\cong \operatorname{Ker}[H_0(F; I_1) \to H_0(F; I_{A\alpha})] \subseteq \operatorname{Ker}[H_0(F; I_1) \to \overline{I(AF)}] \\ &\cong \mathcal{U}(A\tilde{\pi}_{S'})\end{aligned}$$

(applying Proposition 3.3), and similarly for $\operatorname{Coker}(f_S)$. So $\operatorname{Coker}(f_{S'})$ maps trivially to $\operatorname{Coker}(f_S)$. ∎

These results can now be combined to describe $K_2^*(A\pi)$ and $Wh_2^*(A\pi)$. Unfortunately, there does not seem to be any single formula or exact sequence providing a "best" description of these groups. So instead we give several descriptions.

For a p-group π and an unramified p-ring A, define

$$\nu = \nu_{A\pi}\colon \mathscr{U}(A\pi) \to H_2(\pi) \qquad \text{and} \qquad \tilde{\nu} = \tilde{\nu}_{A\pi}\colon \mathscr{U}(A\pi) \to A \otimes H_2(\pi)$$

by setting, for commuting $g, h \in \pi$ and $\lambda \in A$,

$$\nu(h \otimes \lambda g) = \mathrm{Tr}(\lambda) \cdot (h \wedge g), \qquad \tilde{\nu}(h \otimes \lambda g) = \lambda \otimes (h \wedge g)$$

(see Proposition 2.1(iii)). Set

$$\mathscr{U}_0(A\pi) = \mathrm{Ker}[\nu\colon \mathscr{U}(A\pi) \to H_2(\pi)], \qquad \mathscr{U}_0(\pi) = \mathscr{U}_0(\hat{\mathbb{Z}}_p \pi).$$

If $\pi = F/R$ and $\alpha\colon F \to \pi$ is the projection, then

$$I_{A\alpha} = \mathrm{Ker}[AF \to A\pi] \qquad \text{and} \qquad M_{A\alpha} = \mathrm{Ker}[\omega\colon H_0(R; I_{A\alpha}) \to \hat{\mathbb{Z}}_p \otimes R^{ab}]$$

as before. Let

$$\theta\colon \mathscr{U}(A\pi) \xrightarrow{\cong} \mathrm{Ker}[H_0(F; I_{A\alpha}) \to \overline{I(AF)}]$$

again denote the isomorphism of Proposition 3.3.

THEOREM 3.6. *For any p-group π, there are exact sequences*

$$H_3(\pi) \xrightarrow{\eta_{A\pi}} K_2^*(A\pi) \xrightarrow{(\omega_2, \Gamma_2^*)} (A \otimes H_2(\pi)) \oplus \mathscr{U}(A\pi) \xrightarrow[-\tilde{\nu}]{(1-\varphi)\otimes 1} A \otimes H_2(\pi), \tag{1}$$

$$H_3(\pi) \xrightarrow{\eta_{A\pi}} Wh_2^*(A\pi) \xrightarrow{\Gamma_2^*} \mathscr{U}_0(A\pi) \to 0, \tag{2}$$

$$0 \to H_2(\pi) \to K_2^*(A\pi) \to Wh_2^*(A\pi) \to 0. \tag{3}$$

When $A = \hat{\mathbb{Z}}_p$, sequence (3) is split by ω_2.

More precisely, if $\pi = F/R$ and F is free and finitely generated, and $\alpha\colon F \to \pi$ denotes the projection, there is an exact sequence

$$0 \to K_2^*(A\pi) \xrightarrow{(\omega_2, \Gamma_2^*)} (A \otimes H_2(\pi)) \oplus \mathrm{Ker}[\kappa\colon H_0(\pi; M_{A\alpha}) \to \overline{I(AF)}] \xrightarrow{(1-\varphi)\otimes 1 - \xi} A \otimes H_2(\pi), \tag{4}$$

where ξ is the composite

$$\mathrm{Ker}(\kappa) \xrightarrow{\iota/\pi} \mathrm{Ker}[H_0(F; I_{A\alpha}) \to \overline{I(AF)}] \xrightarrow[\cong]{\theta^{-1}} \mathscr{U}(A\pi) \xrightarrow{\tilde{\nu}} A \otimes H_2(\pi)$$

and $\theta^{-1}\circ \imath/\pi\circ \tilde{\Gamma}_2^* = \Gamma_2^*$. Also, $\tilde{\Gamma}_2^*$ reduces to an isomorphism

$$Wh_2^*(A\pi) \xrightarrow{\cong} \mathrm{Ker}[\kappa\colon H_0(\pi; M_{A\alpha}) \to \overline{I(AF)}]. \tag{5}$$

Furthermore, in (1) and (2),

$$\mathrm{Ker}(\eta_{A\pi}) = \mathrm{Im}[\Omega\colon H_1(F; I_\alpha) \to H_1(\pi; R^{ab}) \cong H_3(\pi)], \tag{6}$$

where $I_\alpha = \mathrm{Ker}[\mathbb{Z}F \to \mathbb{Z}\pi]$ and Ω is induced by the map $\omega = \omega_{\mathbb{Z}\alpha}\colon I_\alpha \to R^{ab}$. In particular, $\mathrm{Ker}(\eta_{A\pi})$ is independent of A.

Proof. Let $\mathscr{S}$, and ρ_S, $\tilde{\pi}_S$, α_S for $S \in \mathscr{S}$, be as in Proposition 3.5. The maps $\alpha_S\colon \tilde{\pi}_S \to \pi$ are clearly cofinal among all surjections of p-groups onto π. It follows that

$$\begin{aligned}
K_2^*(A\pi) &\cong \varprojlim_{S\in\mathscr{S}} \mathrm{Coker}[K_2(A\tilde{\pi}_S) \to K_2(A\pi)] \\
&\cong \varprojlim_{S\in\mathscr{S}} \mathrm{Ker}[K_1(A\tilde{\pi}_S, I_S) \to K_1(A\tilde{\pi}_S)] \qquad (I_S = I_{A\alpha_S}) \\
&\cong \varprojlim_{S\in\mathscr{S}} \mathrm{Ker}\left[\frac{K_1(A\tilde{\pi}_S, I_S)}{SK_1(A\rho_S)} \to \frac{K_1(A\tilde{\pi}_S; I(A\tilde{\pi}_S))}{SK_1(A\tilde{\pi}_S)}\right],
\end{aligned}$$

where the last step holds by Proposition 3.5(i).

Hence, by Theorem 2.5 applied to the pairs $\rho_S \lhd \tilde{\pi}_S$ and $\tilde{\pi}_S \lhd \tilde{\pi}_S$, there is an exact sequence

$$\begin{aligned}
0 \to K_2^*(A\pi) \to A\otimes \varprojlim_{S\in\mathscr{S}} \mathrm{Ker}[\rho_S/[\rho_S, \tilde{\pi}_S] \to \tilde{\pi}_S^{ab}] \oplus \varprojlim_{S\in\mathscr{S}} \mathrm{Ker}(\kappa_{A\alpha_S}) \\
\to A\otimes \varprojlim_{S\in\mathscr{S}} \mathrm{Ker}[\rho_S/[\rho_S, \tilde{\pi}_S] \to \tilde{\pi}_S^{ab}].
\end{aligned} \tag{7}$$

But $\varprojlim \mathrm{Ker}(\kappa_{A\alpha_S}) = \mathrm{Ker}(\kappa_{A\alpha})$ by Proposition 3.5, and

$$\varprojlim \mathrm{Ker}[\rho_S/[\rho_S, \tilde{\pi}_S] \to \tilde{\pi}_S^{ab}] \cong \varprojlim \mathrm{Coker}[H_2(\tilde{\pi}_S) \to H_2(\pi)] \cong H_2(\pi).$$

Substituting these into (7) gives sequence (4), where

$$\xi = (\delta^\alpha)^{-1}(\omega^*\imath)/\pi\colon \mathrm{Ker}[\kappa\colon H_0(\pi; M_{A\alpha}) \to \overline{I(AF)}] \to A\otimes H_2(\pi).$$

To see that this equals the composite claimed in the Theorem, we check that the sequence

$$\begin{tikzcd}
\mathrm{Ker}[H_0(F; I_{A\alpha}) \to \overline{I(AF)}] \arrow[r, "\theta^{-1}", "\cong"'] \arrow[d, "\omega^*/\pi"] & \mathscr{U}(A\pi) \arrow[d, "\tilde{\nu}"] \\
A\otimes \mathrm{Ker}[R/[R, F] \to F^{ab}] \arrow[r, "(\delta^\alpha)^{-1}", "\cong"'] & A\otimes H_2(\pi)
\end{tikzcd} \tag{8}$$

commutes. But for any $\lambda \in A$, commuting $g, h \in \pi$, and $\tilde{g} \in \alpha^{-1}g$, $\tilde{h} \in \alpha^{-1}h$,

$$\omega^*\theta(g \otimes \lambda h) = \omega^*(\lambda(\tilde{g}\tilde{h}\tilde{g}^{-1} - \tilde{h})) = \lambda \otimes [\tilde{g}, \tilde{h}] = \delta^\alpha(\tilde{v}(g \otimes \lambda h)).$$

Now consider the exact sequence of Lemma 3.1:

$$H_1(F; I_{A\alpha}) \xrightarrow{\Omega} H_1(\pi; R^{ab}) \to \mathrm{Ker}(\kappa_{A\alpha}) \xrightarrow{\iota/\pi} \mathrm{Ker}[H_0(F; I_{A\alpha}) \to \overline{I(AF)}] \xrightarrow{(\delta^\alpha)^{-1}\omega} H_2(\pi).$$

By the commutativity of (8), the composite

$$\mathscr{U}(A\pi) \xrightarrow{\theta} \mathrm{Ker}[H_0(F; I_{A\alpha}) \to \overline{I(AF)}] \xrightarrow{(\delta^\alpha)^{-1}\omega} H_2(\pi)$$

is equal to v (recall that $\omega = \mathrm{Tr} \circ \omega^*$). Furthermore, the image of $H_1(\pi; R^{ab})$ in $\mathrm{Ker}(\kappa_{A\alpha})$ is contained in $\mathrm{Ker}(\xi)$, and hence in $\mathrm{Im}(\Gamma_2^*)$ in (4) above. So putting this into (4), we get a new exact sequence

$$H_1(\pi; R^{ab}) \xrightarrow{\eta_{A\pi}} K_2^*(A\pi) \xrightarrow{(\omega_2, \Gamma_2^*)} (A \otimes H_2(\pi)) \oplus \mathscr{U}_0(A\pi) \xrightarrow[-\tilde{v}]{(1-\varphi)\otimes 1} A \otimes H_2(\pi),$$

where $\Gamma_2^* = \theta^{-1}(\iota/\pi)\tilde{\Gamma}_2^*$, and $\mathrm{Ker}(\eta_{A\pi}) = \mathrm{Im}(\Omega)$. Sequence (1) now follows upon identifying $H_3(\pi)$ with $H_1(\pi; R^{ab})$ and noting that $(1-\varphi)A \subseteq \mathrm{Ker}(\mathrm{Tr})$ (so that replacing $\mathscr{U}_0(A\pi)$ by $\mathscr{U}(A\pi)$ does not change exactness).

By definition (and the fact that $K_2(A)$ is generated by symbols [13]),

$$Wh_2(A\pi) = \mathrm{Coker}[\lambda_\pi\colon H_2(\pi) \to K_2(A\pi)/\{A^*, A^* \times \pi^{ab}\}].$$

Hence $Wh_2^*(A\pi) \cong K_2^*(A\pi)/\mathrm{Im}(\lambda_\pi)$: the symbols $\{A^*, A^* \times \pi^{ab}\}$ clearly vanish in $K_2^*(A\pi)$. By Theorem 2.5 (and the construction of sequence (4)), the composite

$$H_2(\pi) \xrightarrow{\lambda_\pi} K_2^*(A\pi) \xrightarrow{(\omega_2, \Gamma_2^*)} (A \otimes H_2(\pi)) \oplus \mathrm{Ker}(\kappa_{A\alpha})$$

is inclusion into the first factor. So (3) is exact ($H_2(\pi)$ injects into $K_2^*(A\pi)$), (2) now follows easily from (1), and (5) from (4).

We have seen that $\mathrm{Ker}(\eta_{A\pi}) = \mathrm{Im}(\Omega_{A\alpha})$ where $\Omega_{A\alpha}$ by definition is the composite

$$H_1(F; I_{A\alpha}) \cong A \otimes_{\mathbb{Z}} H_1(F; I_\alpha) \xrightarrow{\mathrm{Tr}\otimes 1} \hat{\mathbb{Z}}_p \otimes_{\mathbb{Z}} H_1(F; I_\alpha) \xrightarrow{\Omega_\alpha} H_1(\pi; R^{ab}) \cong H_3(\pi).$$

Since the trace map is onto and $H_3(\pi)$ is a p-group, this implies that $\mathrm{Im}(\Omega_{A\alpha}) = \mathrm{Im}(\Omega_\alpha)$, and in particular, $\mathrm{Im}(\Omega_{A\alpha})$ is independent of A. ∎

For future reference, we now give explicit descriptions of the maps ω_2 and Γ_2^* occurring in Theorem 3.6.

PROPOSITION 3.7. *The maps*

$$\omega_2(A\pi)\colon K_2^*(A\pi) \to A \otimes H_2(\pi) \qquad \text{and} \qquad \Gamma_2^*(A\pi)\colon K_2^*(A\pi) \to \mathscr{U}(A\pi)$$

are described as follows:

(i) *Let $1 \to \rho \to \tilde{\pi} \to^{\alpha} \pi \to 1$ be any extension of p-groups such that δ^α is injective. Then $\omega_2(A\pi)$ is the composite*

$$K_2(A\pi) \xrightarrow{\partial} \mathrm{Ker}[K_1(A\tilde{\pi}, I_{A\alpha}) \to K_1(A\tilde{\pi})]$$

$$\xrightarrow{\omega^*} A \otimes \mathrm{Ker}[\rho/[\tilde{\pi}, \rho] \to \pi^{ab}] \xleftarrow[\cong]{\delta^\alpha} A \otimes H_2(\pi).$$

(ii) *Let $1 \to \rho \to \tilde{\pi} \to^{\alpha} \pi \to 1$ be any extension of p-groups such that $\alpha(\sigma)$ is cyclic for any abelian $\sigma \subseteq \pi$. Then $\Gamma_2^*(A\pi)$ is the composite*

$$K_2(A\pi) \xrightarrow{\partial} \mathrm{Ker}[K_1(A\tilde{\pi}, I_{A\alpha}) \to K_1(A\tilde{\pi})]$$

$$\xrightarrow{\Gamma_{A\alpha}} \mathrm{Ker}[H_0(\tilde{\pi}, I_{A\alpha}) \to \overline{I(A\tilde{\pi})}] \xleftarrow[\cong]{\theta} \mathscr{U}(A\pi),$$

where $\Gamma_{A\alpha}$ is the map defined in Proposition 2.6.

Proof. These follow directly from Theorem 2.5, Proposition 2.6, and the construction of sequence (1) in Theorem 3.6. ∎

Finally, as a more concrete example, we show:

PROPOSITION 3.8. *Let π be a p-group and A an unramified p-ring. Then, for any $g \in \pi$ and $u \in 1 + I(A[Z_\pi(g)])$,*

$$\Gamma_2^*(\{g, u\}) = g \otimes \Gamma(u) \in \mathscr{U}(A\pi) \qquad (\Gamma = \Gamma_{A[Z_\pi(g)]}).$$

Proof. Let $Z = Z_\pi(g)$. Fix an extension $1 \to \rho \to \tilde{\pi} \to^{\alpha} \pi \to 1$ of p-groups such that $\alpha(\sigma)$ is cyclic for all abelian $\sigma \subseteq \tilde{\pi}$. Lift g to $\tilde{g} \in \tilde{\pi}$ and u to

$$\tilde{u} \in 1 + I(A[\alpha^{-1}Z]).$$

Then $\partial(\{g, u\}) = [\tilde{g}, \tilde{u}] \in K_1(A\tilde{\pi}, I_{A\alpha})$.

Let $G \subseteq \tilde{\pi} \times \tilde{\pi}$ be the pullback over π, and define maps

$$\Delta: A\tilde{\pi} \to AG \qquad \text{and} \qquad \Delta_{\tilde{g}}: A[\alpha^{-1}Z] \to AG$$

by setting $\Delta(h) = (h, h)$, $\Delta_{\tilde{g}}(h) = (\tilde{g}\tilde{h}\tilde{g}^{-1}, \tilde{h})$. Then $[\tilde{g}, \tilde{u}]$ lifts to

$$\Delta_{\tilde{g}}(\tilde{u})\, \Delta(\tilde{u})^{-1} \in \mathrm{Ker}[r_2: K_1(AG) \to K_1(A\tilde{\pi})]$$

in the notation of Proposition 2.6, and so

$$\Gamma_{A\alpha}([\tilde{g}, \tilde{u}]) = r_1(\Delta_{\tilde{g}}(\Gamma(\tilde{u})) - \Delta(\Gamma(\tilde{u}))) = \tilde{g}\Gamma(\tilde{u})\,\tilde{g}^{-1} - \Gamma(\tilde{u}) \in H_0(\tilde{\pi}; I_{A\alpha}).$$

Hence $\Gamma_2^*(\{g, u\}) = g \otimes \Gamma(u)$ by Proposition 3.7 (and the definition of θ in Proposition 3.3). ∎

4

The "mysterious" part of $Wh_2^*(A\pi)$, under the description given in Theorem 3.6, is the group

$$\mathrm{Im}[\eta_{A\pi}: H_3(\pi) \to Wh_2^*(A\pi)] = \mathrm{Ker}[\Gamma_2^*: Wh_2^*(A\pi) \twoheadrightarrow \mathcal{U}_0(A\pi)].$$

This will now be studied; the main results of this section are that $\eta_{A\pi} = 0$ for abelian π, but that $\eta_{A\pi}$ is not zero in general. Recall (Theorem 3.6) that

$$\mathrm{Ker}(\eta_{A\pi}) = \mathrm{Im}[\Omega_\pi: H_1(F; I_\alpha) \to H_1(\pi; R^{ab}) \cong H_3(\pi)]$$

when $\pi = F/R$, $\alpha: F \to \pi$ is the projection, and F is free.

We start with a closer look at $H_1(F; I_\alpha)$.

LEMMA 4.1. *Let $\pi = F/R$ be a p-group, where F is free and $\alpha: F \to \pi$ denotes the projection. Set*

$$I_\alpha = \mathrm{Ker}[\alpha: \mathbb{Z}F \to \mathbb{Z}\pi].$$

Then

$$H_1(F; I_\alpha) \cong \mathrm{Ker}[H_1(F; \mathbb{Z}F) \to H_1(F; \mathbb{Z}\pi)],$$

and is generated by elements of the form $\sum_{i=1}^k a_i \otimes b_i \in H_1(F; \mathbb{Z}F)$ where $a_i, b_i \in F$, and

(i) *$\alpha(b_i) = g$ for some fixed $g \in \pi$ and all i,*

(ii) *$\langle a_i, b_i \rangle \subseteq F$ is cyclic for all i, and*

(iii) *$a_1 a_2 \cdots a_k = 1$ in $(\alpha^{-1}Z_\pi(g))^{ab}$.*

Proof. Let $g_1, \ldots, g_n$ be conjugacy class representatives for π. By Lemma 3.2,

$$H_2(F; \mathbb{Z}\pi) \cong \sum_i H_2(\alpha^{-1}Z_\pi(g_i)) = 0$$

($\alpha^{-1}Z_\pi(g_i) \subseteq F$ is free for all i), and so

$$H_1(F; I_\alpha) \cong \operatorname{Ker}[\alpha_*: H_1(F; \mathbb{Z}F) \to H_1(F; \mathbb{Z}\pi)]$$

by the homology exact sequence.

Applying Lemma 3.2 again, $\operatorname{Ker}(\alpha_*)$ splits as a sum

$$\sum_{j=1}^{n} \operatorname{Ker}[H_1(\alpha^{-1}Z_\pi(g_j); \mathbb{Z}(\alpha^{-1}g_j)) \to H_1(\alpha^{-1}Z_\pi(g_j); \mathbb{Z})].$$

Furthermore, also by Lemma 3.2, $H_1(\alpha^{-1}Z_\pi(g_j); \mathbb{Z}(\alpha^{-1}g_j))$ is generated by elements $a \otimes b$ such that $\alpha(b) = g_j$ and $[a, b] = 1$. In particular, $\langle a, b \rangle$ is an abelian subgroup of F, and hence cyclic. So $\operatorname{Ker}(\alpha_*)$ is generated by elements $\sum_{i=1}^k a_i \otimes b_i$, such that $\alpha(b_i) = g_j$ for all i and some fixed j, such that $\langle a_i, b_i \rangle$ is cyclic for all i, and such that

$$\alpha_*\left(\sum_{i=1}^{k} a_i \otimes b_i\right) = \sum_{i=1}^{k} a_i \otimes g_j = \left(\prod_{i=1}^{k} a_i\right) \otimes g_j = 1 \qquad (1)$$

in $H_1(\alpha^{-1}Z_\pi(g_j); \mathbb{Z}(g_j))$. But (1) holds if and only if

$$a_1 \cdots a_k = 1 \qquad \text{in} \quad (\alpha^{-1}Z_\pi(g_j))^{ab}. \quad ∎$$

In order to make explicit homology calculations, we will use the bar construction as defined, for example, in Section VI.13 of [6], but reflected so as to get a resolution of free *right* modules. In other words, for a group G and left $\mathbb{Z}G$-module M, we let $B'_n(G; M)$ be the sum of terms

$$[g_1 | g_2 | \cdots | g_n] \otimes M$$

for all $(g_1, \ldots, g_n) \in (G-1)^n$. As usual, we set $[g_1 | \cdots | g_n] \otimes M = 0$ if $g_i = 1$ for some i. The boundary maps in lower dimensions are given by

$$\partial([g \mid h] \otimes x) = [g] \otimes hx - [gh] \otimes x + [h] \otimes x$$

and

$$\partial([g] \otimes x) = gx - x$$

for $g, h \in G$ and $x \in M$. In particular, there is the relation

$$[gh] \otimes x \equiv [h] \otimes x + [g] \otimes hx \qquad \text{(mod boundaries)} \qquad (*)$$

in $B'_1(G; M)$. As before, $[g] \otimes x$ will be written $g \otimes x$ for short.

The main result of this section can now be shown:

THEOREM 4.2. *If π is an abelian p-group, then $\eta_{A\pi}=0$ for any A. Hence*

$$Wh_2^*(A\pi)\cong\mathcal{U}_0(A\pi) \qquad \text{and} \qquad K_2^*(\hat{\mathbb{Z}}_p\pi)\cong\mathcal{U}_0(\pi)\oplus H_2(\pi)$$

in this case.

Proof. Write $\pi=\mathbb{Z}/p^{n_1}\times\cdots\times\mathbb{Z}/p^{n_t}$, where $n_1\geqslant n_2\geqslant\cdots\geqslant n_t$, and let $g_1,\dots,g_t$ be the corresponding generators. By the Künneth formula,

$$\begin{aligned}H_3(\pi)&\cong\sum_{i=1}^{t} H_3(\mathbb{Z}/p^{n_i})\oplus\sum_{i<j<k}\mathbb{Z}/p^{n_i}\otimes\mathbb{Z}/p^{n_j}\otimes\mathbb{Z}/p^{n_k}\\&\quad\oplus\sum_{i<j}\operatorname{Tor}(\mathbb{Z}/p^{n_i},\mathbb{Z}/p^{n_j})\\&\cong\pi\oplus\Lambda^2(\pi)\oplus\Lambda^3(\pi)\end{aligned}$$

(where the isomorphisms are, of course, non-canonical).

Let F be the free group on generators $a_1,\dots,a_t$, let $\alpha\colon F\to\pi$ be the map defined by $\alpha(a_i)=g_i$, and set $R=\operatorname{Ker}(\alpha)$. Let

$$\Omega\colon H_1(F;I_\alpha)\to H_1(\pi;R^{ab})\cong H_3(\pi)$$

be the homomorphism of Theorem 3.6; we must show that Ω is onto. To do this, subgroups $L_1,L_2,L_3\subseteq H_1(F;I_\alpha)$ are defined as follows:

$$L_1=\langle x_i=a_i\otimes(a_i^q-1)\colon 1\leqslant i\leqslant t,\ q=p^{n_i}=|g_i|\rangle,$$
$$L_2=\langle y_{ij}=a_ia_j\otimes((a_ia_j)^q-1)\colon 1\leqslant i<j\leqslant t,\ q=p^{n_i}=|g_ig_j|\rangle,$$

and

$$L_3=\langle z_{ijk}=[a_i,a_j]\,a_k\otimes[a_i,a_j]\,a_k-a_k\otimes a_k\colon 1\leqslant i<j<k\leqslant t\rangle.$$

In defining L_3 we are of course identifying

$$H_1(F;I_\alpha)\cong\operatorname{Ker}[a_*\colon H_1(F;\mathbb{Z}F)\to H_1(F;\mathbb{Z}\pi)].$$

Now let $\beta\colon R^{ab}\to R/[F,R]$ be the projection, and

$$\beta_*\colon H_1(\pi;R^{ab})\to H_1(\pi;R/[F,R])\cong\pi\otimes R/[F,R]$$

the induced map. Then for all i and j,

$$\beta_*\Omega(x_i)=g_i\otimes a_i^q\ \ (q=p^{n_i})$$

and

$$\beta_*\Omega(y_{ij})=g_ig_j\otimes(a_ia_j)^q\ \ (q=p^{n_i}).$$

Furthermore, using relation $(*)$ above, we get (for $1 \leqslant i < j < k \leqslant t$)

$$\begin{aligned} z_{ijk} &= [a_i, a_j]\, a_k \otimes ([a_i, a_j] - 1)\, a_k + [a_i, a_j] \otimes a_k \\ &= [a_i, a_j]\, a_k \otimes ([a_i, a_j] - 1)\, a_k \\ &\quad + a_i \otimes (a_j a_i^{-1} a_j^{-1} a_k a_j a_i a_j^{-1} - a_i^{-1} a_j^{-1} a_k a_j a_i) \\ &\quad + a_j \otimes (a_i^{-1} a_j^{-1} a_k a_j a_i - a_j^{-1} a_k a_j) \in H_1(F; I_\alpha), \end{aligned}$$

and hence

$$\begin{aligned} \beta_* \Omega(z_{ijk}) &= g_k \otimes [a_i, a_j] + g_i \otimes [a_j, a_i^{-1} a_j^{-1} a_k a_j a_i] + g_j \otimes [a_i^{-1}, a_j^{-1} a_k a_j] \\ &= g_k \otimes [a_i, a_j] + g_i \otimes [a_j, a_k] + g_j \otimes [a_k, a_i] \in \pi \otimes R/[F, R]. \end{aligned}$$

Since $R/(R \cap [F, F]) \subseteq F^{ab}$ is free abelian, $R/[F, R]$ splits as a sum

$$R/[F, R] \cong \frac{R \cap [F, F]}{[R, F]} \oplus \frac{R}{R \cap [F, F]} \cong H_2(\pi) \oplus \frac{R}{R \cap [F, F]}.$$

Comparison of the elements $\beta_* \Omega(x_i)$, $\beta_* \Omega(y_{ij})$, and $\beta^* \Omega(z_{ijk})$ shows that

$$\beta_* \Omega(L_1 + L_2 + L_3) \cong \pi \oplus \Lambda^2(\pi) \oplus \Lambda^3(\pi) \cong H_3(\pi),$$

and hence that Ω is onto . ∎

We have been unable to get any feel for how "complicated" a group π must be in order that $\eta_{A\pi} \neq 0$. So the examples constructed in the following proposition are probably far from minimal.

PROPOSITION 4.3. *For any prime p there is a p-group π such that $\eta_{A\pi} \neq 0$, i.e., such that $|Wh_2^*(A\pi)| > |\mathcal{U}_0(A\pi)|$, for all A.*

Proof. Fix a prime p, and set $q = p$ if p is odd, $q = 4$ if $p = 2$. Let F be the free group on three generators a, b, c, and define subgroups

$$R = \langle [F, F], a^q, b^q, c^q \rangle$$

and

$$S = \langle [F, R], a^{-q}[b, c], b^{-q}[c, a], c^{-q}[a, b] \rangle.$$

Set $\pi = F/R \cong (\mathbb{Z}/q)^3$ and $\tilde{\pi} = F/S$; there is a central extension

$$1 \to R/S \to \tilde{\pi} \xrightarrow{\beta} \pi \to 1$$

where $R/S \cong (\mathbb{Z}/q)^3$. We will show that $\eta_{A\tilde{\pi}} \neq 0$.

To do this, we define another subgroup

$$T = \langle [F, R], a^q, b^q, c^q, [a, b]^p, [a, c]^p, [b, c]^p \rangle,$$

set $G = F/T$, and consider the central extension

$$1 \to R/T \to G \xrightarrow{\gamma} \pi \to 1 \qquad (R/T \cong (\mathbb{Z}/p)^3).$$

Note in particular:

(i) any cyclic subgroup of π lifts to G: for example,

$$(ab)^q = a^q b^q [a, b]^{q(q-1)/2} = 1 \qquad \text{in} \quad G \ (p \mid \tfrac{1}{2}q(q-1));$$

(ii) if $p = 2$, then any subgroup of π ($\cong (\mathbb{Z}/4)^3$) isomorphic to $\mathbb{Z}/4 \times \mathbb{Z}/2 \times \mathbb{Z}/2$ lifts to G: any liftings of elements of orders two and four in π commute in G.

We now claim that

$$\beta_*(\operatorname{Ker}(\eta_{A\tilde{\pi}})) \subseteq \operatorname{Im}[\gamma_*: H_3(G) \to H_3(\pi)]. \tag{1}$$

Let $\alpha: F \to \pi$ and $\tilde{\alpha}: F \to \tilde{\pi}$ be the projections, and identify

$$H_1(F; I_{\tilde{\alpha}}) \cong \operatorname{Ker}[H_1(F; \mathbb{Z}F) \to H_1(F; \mathbb{Z}\tilde{\pi})]$$

as in Lemma 4.1. Fix $g \in \tilde{\pi}$ and $\sum_{i=1}^k a_i \otimes b_i \in H_1(F; I_{\tilde{\alpha}})$, where $a_i, b_i \in F$, $\alpha(b_i) = g$, and $\langle a_i, b_i \rangle \subseteq F$ is cyclic for all i, and $a_1 \cdots a_k = 1$ in $(\alpha^{-1}Z)^{ab}$ ($Z = Z_{\tilde{\pi}}(g)$). Then

$$\Omega_{\tilde{\pi}}\left(\sum a_i \otimes b_i\right) \in \operatorname{Im}[H_3(Z) \to H_3(\tilde{\pi})],$$

and so

$$\beta_* \Omega_{\tilde{\pi}}\left(\sum a_i \otimes b_i\right) \in \operatorname{Im}[H_3(\beta Z) \to H_3(\pi)].$$

If $g \notin R/S$, then $\beta(Z) = \beta(Z_{\tilde{\pi}}(g))$ is either cyclic or isomorphic to $\mathbb{Z}/4 \times (\mathbb{Z}/2)^2$, and hence lifts to G by (i) or (ii) above. So in this case,

$$\beta_* \Omega_{\tilde{\pi}}\left(\sum a_i \otimes b_i\right) \in \operatorname{Im}[H_3(G) \to H_3(\pi)].$$

Now assume $g \in R/S$. Then $b_i \in R$ for all i, so

$$\sum a_i \otimes b_i = \sum a_i \otimes (b_i - 1) \in H_1(F; I_\alpha) \cong \operatorname{Ker}[H_1(F; \mathbb{Z}F) \to H_1(F; \mathbb{Z}\pi)],$$

where now $(b_i-1)\in I_\alpha$ for all i. Then

$$\beta_*\Omega_{\tilde\pi}\left(\sum a_i\otimes b_i\right)=\Omega_\pi\left(\sum a_i\otimes(b_i-1)\right)=\sum a_i\otimes b_i\in H_1(\pi;R^{ab})\cong H_3(\pi).$$

Since $\langle a_i, b_i\rangle\subseteq F$ is cyclic for each i, each term $a_i\otimes b_i$ is a homology element, and in fact lies in the image of $H_3(\sigma_i)$ for some cyclic $\sigma_i\subseteq\pi$. By (i), each σ_i lifts to G, and so again

$$\beta_*\Omega_{\tilde\pi}\left(\sum a_i\otimes b_i\right)\in\mathrm{Im}[H_3(G)\to H_3(\pi)].$$

Hence (1) follows upon combining these two cases:

$$\beta_*(\mathrm{Ker}(\eta_{A\tilde\pi}))=\beta_*\Omega_{\tilde\pi}(H_1(F;I_{\tilde\alpha}))\subseteq\mathrm{Im}[\gamma_*\colon H_3(G)\to H_3(\pi)].$$

Finally, consider the element

$$\begin{aligned}x=bab^{-1}\otimes[b^{-1},bcb^{-1}](bab^{-1})^q+cbc^{-1}\otimes[c^{-1},cac^{-1}](cbc^{-1})^q\\ +aca^{-1}\otimes[a^{-1},aba^{-1}](aca^{-1})^q\in H_1(\tilde\pi;S^{ab})\cong H_3(\tilde\pi).\end{aligned}\tag{2}$$

That x actually is a cycle follows from the Hall commutator identity

$$^b[a,[b^{-1},c]]\cdot{}^c[b,[c^{-1},a]]\cdot{}^a[c,[a^{-1},b]]=1\qquad\text{in } F.$$

We claim that $\beta_*(x)\notin\mathrm{Im}(\gamma_*)$, and hence by (1) that $x\notin\mathrm{Ker}(\eta_{A\tilde\pi})$. Identify

$$H_3(G)\cong H_1(G;T^{ab})\qquad\text{and}\qquad H_3(\pi)\cong H_1(\pi;R^{ab});$$

then

$$\mathrm{Im}[\gamma_*\colon H_1(G;T^{ab})\to H_1(\pi;R^{ab})]\subseteq\mathrm{Ker}[\partial\colon H_1(\pi;R^{ab})\to\pi\otimes R/T].$$

Upon simplifying (2), we get

$$\partial(\beta_*x)=-a\otimes[b,c]-b\otimes[c,a]-c\otimes[a,b]\in\pi\otimes R/T$$

(recall that $a^q, b^q, c^q\in T$). But $\{[b,c],[c,a],[a,b]\}$ is a basis for $R/T\cong(\mathbb{Z}/p)^3$, so $\partial(\beta_*x)\neq 0$ and $\beta_*(x)\notin\mathrm{Im}(\gamma_*)$. ∎

To finish the section, we show:

PROPOSITION 4.4. *For any prime p, unramified p-ring A, and p-group π, $K_2^*(A\pi)=0$ if and only if π is cyclic or quaternionic.*

Proof. Assume first that π is not cyclic or quaternionic. Fix any $b\in Z(\pi)$ of order p. Using Theorem 5.4.10 in [5], choose any $a\notin\langle b\rangle$ in π

also of order p. Choose $g \in \pi$ such that $g^{p^k} = a$ for some k, but such that g is not a p-power in π. Then

$$S_\pi(g) = \langle \sigma \subseteq \pi \mid \sigma \text{ cyclic}, g \in \sigma \rangle = \langle g \rangle.$$

So $b \in Z_\pi(g) - S_\pi(g)$, $(Z_\pi(g)/S_\pi(g))^{ab} \neq 1$, and so $\mathscr{U}(\pi) \neq 1$. But

$$\mathscr{U}_0(A\pi) = \mathrm{Ker}[\mathrm{Tr} \otimes \nu \colon A \otimes \mathscr{U}(\pi) \to H_2(\pi)],$$

and so by Theorem 3.6

$$|K_2^*(A\pi)| \geqslant |H_2(\pi)| \cdot |\mathscr{U}_0(A\pi)| \geqslant |A \otimes \mathscr{U}(\pi)| > 1.$$

If π is cyclic, then $K_2^*(A\pi) = 0$ by Theorem 3.6 and 4.2 (or just from the fact that $K_2(A\pi)$ is generated by symbols). If π is quaternionic, then $H_2(\pi) = 0$ (see Section XII.7 of [3]), and one easily checks that $\mathscr{U}(\pi) = 0$. So by Theorem 3.6, it remains only to check that $\eta_{A\pi} = 0$ in this case.

Set $2^n = |\pi|$ $(n \geqslant 3)$, and write $\pi = F/R$ where $F = \langle a, b \rangle$ and R is the normal subgroup generated by

$$\{a^{2^{n-1}}, a^{2^{n-2}}b^{-2}, bab^{-1}a\}.$$

Let $\alpha \colon F \to \pi$ be the projection. Consider the element

$$\begin{aligned} x = a \otimes a^{2^{n-2}} + b \otimes b^2 - ab \otimes (ab)^2 &\in \mathrm{Ker}[H_1(F; \mathbb{Z}F) \to H_1(F; \mathbb{Z}\pi)] \\ &\cong H_1(F; I_\alpha) \end{aligned}$$

(see Lemma 4.1). Modulo boundaries,

$$\begin{aligned} x &= a \otimes a^{2^{n-2}} + b \otimes b^2 - b \otimes (ab)^2 - a \otimes b(ab)^2 b^{-1} \\ &= a \otimes (a^{2^{n-2}} - (ba)^2) + b \otimes (b^2 - (ab)^2) \end{aligned}$$

(see relation $(*)$ above), and so

$$\Omega(x) = a \otimes a^{2^{n-2}}(ba)^{-2} + b \otimes b^2(ab)^{-2} \in H_1(\pi; R^{ab}) \cong H_3(\pi).$$

Now let

$$\begin{aligned} \beta_* \colon H_1(\pi; R^{ab}) \to H_1\left(\pi; \frac{R}{R \cap [F, F]}\right) &\cong \pi^{ab} \otimes \frac{R}{R \cap [F, F]} \\ &\cong (\mathbb{Z}/2)^2 \otimes \langle a^2, b^2 \rangle \end{aligned}$$

be the projection. Then

$$\beta_* \Omega(x) = a \otimes a^{2^{n-2}-2}b^{-2} + b \otimes a^{-2}$$

is not a multiple of two in $\pi^{ab}\otimes R/(R\cap[F,F])$. So $\Omega(x)$ must generate $H_3(\pi)\cong\mathbb{Z}/2^n$ [3, Sect. XII.7], Ω is onto, and hence $\eta_{A\pi}=0$ by Theorem 3.6(5). ∎

5

These results will now be applied to get lower bounds on the size of $K_2(\mathbb{Z}\pi)$ and $Wh_2(\pi)$ for p-groups π. The main problem is to get an appropriate localization sequence for relating $K_2(\mathbb{Z}\pi)$ to $K_2^*(\hat{\mathbb{Z}}_p\pi)$.

THEOREM 5.1. *Let $\mathfrak{A}\subseteq\mathfrak{B}$ be $\mathbb{Z}$-orders in a semisimple $\mathbb{Q}$-algebra A. Then, for any prime p, there is an exact sequence*

$$0\to\varprojlim_n\operatorname{Coker}[K_2(\mathfrak{A})\to K_2(\mathfrak{A}/p^n\mathfrak{A})]\to\varprojlim_n\operatorname{Coker}[K_2(\mathfrak{B})\to K_2(\mathfrak{B}/p^n\mathfrak{B})]$$
$$\to Cl_1(\mathfrak{A})_{(p)}\to Cl_1(\mathfrak{B})_{(p)}\to 0.$$

Proof. This follows from Proposition 1.2 in [11], except for showing that the map

$$\varphi\colon\varprojlim_n\operatorname{Coker}[K_2(\mathfrak{A})\to K_2(\mathfrak{A}/p^n\mathfrak{A})]\to\varprojlim_n\operatorname{Coker}[K_2(\mathfrak{B})\to K_2(\mathfrak{B}/p^n\mathfrak{B})]$$

is injective.

Step 1. We first show that the map

$$\varphi_0\colon\varprojlim_I K_1(\mathfrak{A},I)\to\varprojlim_I K_1(\mathfrak{B},I)$$

is injective, where the limits are taken over all ideals of finite index. It suffices to show for any $\mathfrak{B}$-ideal $I\subseteq\mathfrak{A}$ of finite index that there exists $I'\subseteq I$ such that the map

$$\operatorname{Ker}[K_1(\mathfrak{A},I')\to K_1(\mathfrak{B},I')]\to\operatorname{Ker}[K_1(\mathfrak{A},I)\to K_1(\mathfrak{B},I)]$$

is zero. By stability results [2, Corollary I.4.5], this means finding I' such that (in $GL_3(\mathfrak{B})$)

$$E_3(\mathfrak{B},I')\subseteq E_3(\mathfrak{A},I).$$

Recall that $\operatorname{Ker}[K_1(\mathfrak{A},I)\to K_1(\mathfrak{B})]$ is finite (see Corollary X.3.6 in [2]). Clearly $GL_3(\mathfrak{A},I)$ has finite index in $GL_3(\mathfrak{B})$, and it follows that $E_3(\mathfrak{A},I)$ has finite index in $E_3(\mathfrak{B})$. So there is a normal subgroup $G\lhd E_3(\mathfrak{B})$ of finite index and contained in $E_3(\mathfrak{A},I)$. Choose I' small enough so that $e_{ij}^r\in G$ for all $r\in I'$ and $1\leqslant i,j\leqslant 3$ (e.g., take $I'\subseteq e\mathfrak{B}$ where $e=\exp(E_3(\mathfrak{B})/G)$). Since

$E_3(\mathfrak{B}, I')$ is the smallest normal subgroup of $E_3(\mathfrak{B})$ containing all such e_{ij}^r, we have the required relation:

$$E_3(\mathfrak{B}, I') \subseteq G \subseteq E_3(\mathfrak{A}, I).$$

Step 2. The exact sequence for an ideal induces an injection

$$\varprojlim_I \operatorname{Coker}[K_2(\mathfrak{A}) \to K_2(\mathfrak{A}/I)] \to \varprojlim_I K_1(\mathfrak{A}, I)$$

(where limits are still taken over all ideals of finite index), and similarly for $\mathfrak{B}$. Hence, by Step 1, the map

$$\varphi_1 \colon \varprojlim_I \operatorname{Coker}[K_2(\mathfrak{A}) \to K_2(\mathfrak{A}/I)] \to \varprojlim_I \operatorname{Coker}[K_2(\mathfrak{B}) \to K_2(\mathfrak{B}/I)]$$

is injective.

Recall that for any finite ring R, $K_2(R)$ is finite with torsion only for primes dividing $|R|$ (see, e.g., Theorem 2.4 in [11]). In particular, for any $I \subseteq \mathfrak{A}$ of finite index,

$$\begin{aligned} \operatorname{Coker}[K_2(\mathfrak{A}) \to K_2(\mathfrak{A}/I)] \cong{}& \operatorname{Coker}[K_2(\mathfrak{A}) \to K_2((\mathfrak{A}/I)_{(p)})] \\ &\oplus \operatorname{Coker}[K_2(\mathfrak{A}) \to K_2((\mathfrak{A}/I)[1/p])] \end{aligned}$$

(and similarly for $\mathfrak{B}$). So φ_1 splits as a corresponding sum of injections. But

$$\varprojlim_I \operatorname{Coker}[K_2(\mathfrak{A}) \to K_2((\mathfrak{A}/I)_{(p)})] \cong \varprojlim_n \operatorname{Coker}[K_2(\mathfrak{A}) \to K_2(\mathfrak{A}/p^n\mathfrak{A})],$$

and so the map φ above is injective. ∎

Based on the notation in [1], we now set

$$\begin{aligned} C(\Sigma, A) &= \varprojlim_I \operatorname{Coker}[K_2(\mathfrak{M}) \to K_2(\mathfrak{M}/I)] \\ &\cong \prod_p \varprojlim_n \operatorname{Coker}[K_2(\mathfrak{M}) \to K_2(\mathfrak{M}/p^n\mathfrak{M})] \end{aligned}$$

for any semisimple $\mathbb{Q}$-algebra A and maximal $\mathbb{Z}$-order $\mathfrak{M} \subseteq A$. By Theorem 5.2 in [1], $C(\Sigma, A)$ is finite for any A. Using this, we now get

THEOREM 5.2. *Let π be a p-group, and R the ring of integers in some finite extension K of $\mathbb{Q}$ where p is unramified. Set*

$$C^*(\Sigma, K\pi) = C(\Sigma, K\pi)_{(p)} / \langle \{a, u\} \in C(\Sigma, K\pi) \mid a \in (\hat{R}_p)^*, u \in (\hat{R}_p\pi)^* \rangle,$$

and let $r = rk_{\mathbb{Z}}(R)$. Then

$$|K_2(R\pi)_{(p)}| \geq \frac{|\mathscr{U}(\pi)|^r \cdot |SK_1(R\pi)|}{|C^*(\Sigma, K\pi)|}$$

and

$$|Wh_2(\pi)_{(p)}| \geqq \frac{|\mathcal{U}(\pi)| \cdot |SK_1(\mathbb{Z}\pi)|}{|H_2(\pi)| \cdot |C^*(\Sigma, \mathbb{Q}\pi)|}.$$

Proof. Let $\mathfrak{M} \supseteqq R\pi$ be a maximal order in $K\pi$; $Cl_1(\mathfrak{M})=0$ by [7]. Applying Theorem 5.1 to the pair $R\pi \subseteqq \mathfrak{M}$, we get

$$|\varprojlim_n \text{Coker}[K_2(R\pi) \to K_2(R/p^n[\pi])]| = |C(\Sigma, K\pi)_{(p)}| / |Cl_1(R\pi)|. \quad (1)$$

Furthermore,

$$\varprojlim_n \text{Coker}[K_2(R\pi) \to K_2(R/p^n[\pi])] \cong \text{Coker}[K_2(R\pi) \to K_2^{\text{top}}(\hat{R}_p\pi)]$$

if $K_2(R\pi)$ is finite, and so in any case

$$|K_2(R\pi)_{(p)}| \geqq |K_2^{\text{top}}(\hat{R}_p\pi)| / |\varprojlim_n \text{Coker}[K_2(R\pi) \to K_2(R/p^n[\pi])]|. \quad (2)$$

By Proposition 0.1, $K_2^{\text{top}}(\hat{R}_p\pi)$ maps onto $K_2^*(\hat{R}_p\pi)$, and symbols of the form $\{a, u\}$ for $a \in (\hat{R}_p)^*$ and $u \in (\hat{R}_p\pi)^*$ clearly lie in the kernel. It follows that

$$|K_2^{\text{top}}(\hat{R}_p\pi)| \geqq |K_2^*(\hat{R}_p\pi)| \cdot |\langle \{a, u\} \in C(\Sigma, K\pi)_{(p)}: a \in (\hat{R}_p), u \in (\hat{R}_p\pi)^* \rangle|. \quad (3)$$

Write $\hat{R}_p = \sum A_i$, a sum of unramified p-rings. For each i,

$$\mathcal{U}_0(A_i\pi) = \text{Ker}[v: \mathcal{U}(A_i\pi) \to H_2(\pi)],$$

where

$$\text{Im}(v) = \langle g \wedge h: g, h \in \pi, gh = hg \rangle = H_2^{ab}(\pi)$$

(see Section 3 in [10]). Hence by Theorem 3.6,

$$\begin{aligned} |K_2^*(A_i\pi)| \geqq |\mathcal{U}_0(A_i\pi)| \cdot |H_2(\pi)| &= |A_i \otimes \mathcal{U}(\pi)| \cdot |H_2(\pi)| / |H_2^{ab}(\pi)| \\ &= |A_i \otimes \mathcal{U}(\pi)| \cdot |SK_1(A_i\pi)| \end{aligned}$$

(also applying Theorem 3 in [10]), and so

$$|K_2^*(\hat{R}_p\pi)| \geqq |\hat{R}_p \otimes \mathcal{U}(\pi)| \cdot |SK_1(\hat{R}_p\pi)| = |\mathcal{U}(\pi)|^r \cdot |SK_1(\hat{R}_p\pi)|. \quad (4)$$

Combining (1), (2), (3), and (4) now gives the formula

$$|K_2(R\pi)_{(p)}| \geqq |\mathcal{U}(\pi)|^r \cdot |SK_1(R\pi)| / |C^*(\Sigma, K\pi)|.$$

To obtain the second inequality, we must check that

$$|K_2^{\mathrm{top}}(\hat{\mathbb{Z}}_p\pi)/\langle\{-1,-1\},\{-1,g\}: g\in\pi\rangle|$$
$$\geqq |K_2^*(\hat{\mathbb{Z}}_p\pi)|\cdot|\langle\{a,u\}\in C(\Sigma,\mathbb{Q}\pi)_{(p)}\rangle|.$$

If p is odd, this follows directly from (3). If $p=2$, then the symbols $\{-1,-1\}$ and $\{-1,g\}$ are clearly trivial in $K_2^*(\hat{\mathbb{Z}}_p\pi)$, and we must check that they also map trivially to $C(\Sigma,\mathbb{Q}\pi)$. It suffices to check this when π is cyclic. But $C(\Sigma,\mathbb{Q})=0$, and for any $n\geqq 2$ and any primitive 2^n-th root of unity ζ,

$$\{-1,\zeta\}=1 \qquad \text{in} \quad K_2^{\mathrm{top}}(\hat{\mathbb{Z}}_2[\zeta])\cong C(\Sigma,\mathbb{Q}\zeta)_{(2)}$$

(this follows easily from Theorem 15.7 in [9]). ∎

For abelian p-groups, for example, more explicit lower bounds for $|K_2(\mathbb{Z}\pi)|$ can easily be derived from those in Theorem 5.3, generalizing the formulas in [4].

6

We now study the groups

$$K_2^*(A/p^n[\pi]) = \varprojlim_{\tilde{\pi}\twoheadrightarrow\pi} \operatorname{Coker}[K_2(A/p^n[\tilde{\pi}])\to K_2(A/p^n[\pi])]$$

when π is a p-group, A an unramified p-ring, and $n\geqslant 1$. A description of them will be given analogous to the description of $K_2^*(A\pi)$ in Theorem 3.6. This will then be used to show that $K_2^*(\mathbb{F}_q\pi)=K_2(\mathbb{F}_q\pi)$ when π is an abelian p-group (and $q=p^f$), thus providing a complete description of $K_2(\mathbb{F}_q\pi)$ in this case.

The relationship between $K_2^*(A/p^n[\pi])$ and $K_2^*(A\pi)$ is described by:

Proposition 6.1. *For any p-group π and unramified p-ring A, define a homomorphism*

$$\psi: \mathscr{U}(A\pi)\to K_2^*(A\pi)$$

by setting $\psi(h\otimes\lambda g)=\{h,\exp(p\lambda(g-1))\}$ for $\lambda\in A$ and commuting $g,h\in\pi$. Then for any $n\geqslant 1$, the sequence

$$\mathscr{U}(A\pi)\xrightarrow{p^{n-1}\psi} K_2^*(A\pi)\to K_2^*(A/p^n[\pi])\to 0 \tag{1}$$

is exact. Furthermore, in the notation of Theorem 3.6,

$$\omega_2\psi(h\otimes\lambda g)=p\lambda(h\wedge g)=p\tilde{\nu}(h\otimes\lambda g)\in A\otimes H_2(\pi)$$

and (*letting* $\varphi\in\mathrm{Gal}(A/\hat{\mathbb{Z}}_p)$ *denote the Frobenius automorphism*)

$$\Gamma_2^*\psi(h\otimes\lambda g)=h\otimes p\lambda g-h\otimes\varphi(\lambda)\,g^p\in\mathscr{U}(A\pi).$$

Proof. Note first that ψ is well defined: if $g\in\pi$, $\lambda\in A$, and $n\in\mathbb{Z}$, then

$$\{g,\exp(p\lambda(g^n-1))\}=1\qquad\text{in}\quad K_2^*(A\pi)$$

(the symbol lifts to arbitrary large p-groups $\tilde{\pi}\twoheadrightarrow\pi$). Clearly,

$$\mathrm{Im}(p^{n-1}\psi)\subseteq\mathrm{Ker}[K_2^*(A\pi)\to K_2^*(A/p^n[\pi])].$$

To show inclusion the other way, choose an extension

$$1\to\rho\to\tilde{\pi}\xrightarrow{\alpha}\pi\to 1$$

of p-groups such that

$$K_2^*(A\pi)=\mathrm{Coker}[K_2(A\tilde{\pi})\to K_2(A\pi)],$$
$$K_2^*(A/p^n[\pi])=\mathrm{Coker}[K_2(A/p^n[\tilde{\pi}])\to K_2(A/p^n[\pi])].$$

Let $\mathfrak{A}\subseteq A\tilde{\pi}\times A\tilde{\pi}$ be the pullback over $A\pi$, as usual, and let $r_i\colon\mathfrak{A}\to A\tilde{\pi}$ denote the projections. Then $\mathfrak{A}/p^n\mathfrak{A}$ is the pullback of $A/p^n[\tilde{\pi}]$ with itself over $A/p^n[\pi]$, and so there are Mayer–Vietoris sequences

$$\begin{array}{ccccccc}
0\to & K_2^*(A\pi) & \xrightarrow{\partial} & K_1(\mathfrak{A}) & \xrightarrow{r_{1*}\oplus r_{2*}} & K_1(A\tilde{\pi})\oplus K_1(A\tilde{\pi}) \\
& \downarrow & & \downarrow & & \downarrow \\
0\to & K_2^*(A/p^n[\pi]) & \longrightarrow & K_1(\mathfrak{A}/p^n\mathfrak{A}) & \longrightarrow & K_1(A/p^n[\tilde{\pi}])\oplus K_1(A/p^n[\tilde{\pi}])
\end{array}$$

Hence, for any

$$\xi\in\mathrm{Ker}[K_2^*(A\pi)\to K_2^*(A/p^n[\pi])],$$

we have

$$\partial(\xi)\in\mathrm{Ker}[K_1(\mathfrak{A})\to K_1(\mathfrak{A}/p^n\mathfrak{A})]=\mathrm{Im}[K_1(\mathfrak{A},p^n\mathfrak{A})\to K_1(\mathfrak{A})].$$

Write $\partial(\xi)=[1+p^nx]$ for some $x\in\mathfrak{A}$. Since r_2 is split by the diagonal map $\Delta\colon A\tilde{\pi}\to\mathfrak{A}$, we may assume that $x\in\mathrm{Ker}(r_2)$, and hence $r_1(x)\in\mathrm{Ker}(\alpha)$.

Set

$$p^n y = \log(1 + p^n \cdot r_1(x)) \qquad (y \in \mathrm{Ker}[\alpha\colon A\tilde{\pi} \to A\pi]).$$

Then $1 + p^n \cdot r_1(x) = \exp(p^n y)$: this is clear if $p^n > 2$, and holds when $p^n = 2$ since $r_1(x) \in \mathrm{Ker}(\alpha) \subseteq J(A\tilde{\pi})$ (and thus $x^k \to 0$ as $k \to \infty$). In other words,

$$[\exp(p^n y)] = r_{1*}(\partial(\xi)) \in K_1(A\tilde{\pi}, \mathrm{Ker}(\alpha)). \tag{2}$$

Write $y = \sum_{k \in \pi} y_k$, where $y_k \in A(\alpha^{-1}k)$ for $k \in \pi$. Then $y_k \in \mathrm{Ker}(\alpha)$ for all k, and $\exp(p^n y_k) = \exp(p^n g y_k g^{-1})$ in $K_1(A\tilde{\pi}, \mathrm{Ker}(\alpha))$ for any $g \in \tilde{\pi}$. So after conjugating, we may assume that $y = \sum y_{k_i}$, where the k_i run through a set of conjugacy class representatives for π. Since $[\exp(p^n y)] = 1$ in $K_1(A\tilde{\pi})$, we must have $y = 0$ in $\overline{A\tilde{\pi}}$, and hence each y_{k_i} is a sum of terms of the form $(hgh^{-1} - g)$ for $g, h \in \tilde{\pi}$ such that $[\alpha(g), \alpha(h)] = 1$ in π.

So we can now write

$$y = \sum_i \lambda_i(h_i g_i h_i^{-1} - g_i) \qquad (\lambda_i \in A,\ g_i, h_i \in \tilde{\pi},\ [\alpha g_i, \alpha h_i] = 1).$$

Then by (2):

$$\begin{aligned}\partial(\xi) &= \partial\left(\prod_i \{\alpha(h_i), \exp(p^n \lambda_i(\alpha(g_i) - 1))\}\right)\\ &= p^{n-1}\, \partial\psi\left(\sum_i \alpha(h_i) \otimes \lambda_i \alpha(g_i)\right).\end{aligned}$$

Since ∂ is one-to-one, this shows that $\xi \in \mathrm{Im}(p^{n-1}\psi)$.

Thus, sequence (1) is exact at $K_2^*(A\pi)$. Exactness at $K_2^*(A/p^n[\pi])$ follows since $K_2(A\pi)$ surjects onto $K_2(A/p^n[\pi])$ by Lemma 1.7 in [11] (or, when $p^n = 2$, since the only torsion in $K_1(A\pi, 2A\pi)$ is -1 by Proposition 2 in [10]). The descriptions of $\omega_2\psi$ and $\Gamma_2^*\psi$ follow immediately from Propositions 3.7 and 3.8 (and the definition of Γ in [10]). ▮

There are several ways of combining Proposition 6.1 with Theorem 3.6 to get descriptions of $K_2^*(A/p^n[\pi])$; we give only one of them here. Note in particular that $H_2(\pi)$ still injects into $K_2^*(A/p^n[\pi])$.

PROPOSITION 6.2. *For any p-group π, any unramified p-ring A, and any $n \geqslant 1$, there is a short exact sequence*

$$0 \longrightarrow H_2(\pi) \to \frac{K_2^*(A/p^n[\pi])}{\mathrm{Im}(\bar{\eta})}$$

$$\xrightarrow{\Gamma_2^*} \frac{\mathscr{U}_0(A\pi)}{\langle h \otimes p^n \lambda g - h \otimes p^{n-1} \varphi(\lambda)\, g^p \rangle} \to 0.$$

Here $\bar{\Gamma}_2^*$ is the reduction of Γ_2^*, and $\bar{\eta}$ denotes the composite

$$\bar{\eta}\colon H_3(\pi) \xrightarrow{\eta_{A\pi}} K_2^*(A\pi) \to K_2^*(A/p^n[\pi]).$$

Proof. Let $\psi\colon \mathscr{U}(A\pi) \to K_2^*(A\pi)$ be the map defined above. By Proposition 6.1 and Theorem 3.6, (ω_2, Γ_2^*) induces an injection

$$\frac{K_2^*(A/p^n[\pi])}{\mathrm{Im}(\bar{\eta})} \rightarrowtail \frac{(A \otimes H_2(\pi)) \oplus \mathscr{U}(A\pi)}{p^{n-1}(\omega_2\psi, \Gamma_2^*\psi)(\mathscr{U}(A\pi))}.$$

Then (using Theorem 3.6(1) again)

$$\begin{aligned}
&\mathrm{Ker}\left[\bar{\Gamma}_2^*\colon \frac{K_2(A/p^n[\pi])}{\mathrm{Im}(\bar{\eta})} \to \frac{\mathscr{U}(A\pi)}{p^{n-1}\Gamma_2^*\psi(\mathscr{U}(A\pi))}\right] \\
&\quad\cong \frac{\omega_2(\mathrm{Ker}(\Gamma_2^*))}{\omega_2(\mathrm{Ker}(\Gamma_2^*) \cap \mathrm{Im}(\psi))} \\
&\quad= \frac{\mathrm{Ker}[(1-\varphi)\otimes 1\colon A \otimes H_2(\pi) \to A \otimes H_2(\pi)]}{\omega_2\psi(\mathrm{Ker}(\Gamma_2^*\psi))} = \frac{H_2(\pi)}{\omega_2\psi(\mathrm{Ker}(\Gamma_2^*\psi))}.
\end{aligned}$$

Furthermore (see Theorem 3.6(2))

$$\mathrm{Im}(\bar{\Gamma}_2^*) = \frac{\mathrm{Im}(\Gamma_2^*)}{p^{n-1}\Gamma_2^*\psi(\mathscr{U}(A\pi))} = \frac{\mathscr{U}_0(A\pi)}{\langle h \otimes p^n\lambda g - h \otimes p^{n-1}\varphi(\lambda)\, g^p \rangle}.$$

So we will be done upon showing that $\mathrm{Ker}(\Gamma_2^*\psi) \subseteq \mathrm{Ker}(\omega_2\psi)$.

Define

$$\Phi\colon \mathscr{U}(A\pi) \to \mathscr{U}(A\pi)$$

by setting $\Phi(h \otimes \lambda g) = h \otimes \varphi(\lambda)\, g^p$. Then

$$\Gamma_2^*\psi(x) = px - \Phi(x)$$

for any $x \in \mathscr{U}(A\pi)$. Fix

$$x = \sum_{i=1}^{k} x_i \in \mathrm{Ker}(\Gamma_2^*\psi),$$

where for each i, x_i is the sum of terms $h \otimes \lambda g$ with $|g| = p^i$ (note that $h \otimes \lambda 1 = 0$ in $\mathscr{U}(A\pi)$). Then

$$\begin{aligned}
0 = \Gamma_2^*\psi(x) &= \sum_{i=1}^{k} (px_i - \Phi(x_i)) \\
&= px_k + \sum_{i=1}^{k-1} (px_i - \Phi(x_{i+1})),
\end{aligned}$$

so that $px_k = 0$ and $px_i = \Phi(x_{i+1})$ for all $i < k$. Hence, for all i,

$$\tilde{v}(px_i) = \tilde{v}(\Phi(x_{i+1})) = (\varphi \otimes 1)(\tilde{v}(px_{i+1})) \in A \otimes H_2(\pi),$$

and by induction

$$\tilde{v}(px_i) = (\varphi^{k-i} \otimes 1)(\tilde{v}(px_k)) = 0.$$

Thus, $\omega_2 \psi(x) = p\tilde{v}(x) = 0$, and we are done. ∎

Note that the short exact sequence of Proposition 6.2 is *not* split by ω_2 when $A = \hat{\mathbb{Z}}_p$, since $\omega_2 \psi \neq 0$ in general.

For the rest of this section, we restrict attention to abelian p-groups. We will show that $K_2(\mathbb{F}_q \pi) = K_2^*(\mathbb{F}_q \pi)$ for abelian π (and $q = p^f$) by first computing $|K_2^*(\mathbb{F}_q \pi)|$; and then showing that $|K_2(\mathbb{F}_q \pi)|$ can be at most as large.

For any abelian p-group π and any $i \geq 1$, we write

$$p^i\text{-}rk(\pi) = \dim_{\mathbb{F}_p}(\pi^{p^{i-1}}/\pi^{p^i}),$$

i.e., the number of cyclic summands in the decomposition of π of order at least p^i. Also, $\operatorname{ord}_p(|\pi|)$ is used to denote the number n such that $|\pi| = p^n$.

Proposition 6.3. *For any abelian p-group π, any unramified p-ring A with $f = [A: \hat{\mathbb{Z}}_p]$, and any $n \geq 1$,*

$$|K_2^*(A/p^n[\pi])| = |\mathscr{U}(\pi) \otimes \mathbb{Z}/p^n|^f.$$

If $\exp(\pi) = p^e$ *and* $r_i = p^i\text{-}rk(\pi)$ $(1 \leq i \leq e)$, *then*

$$\begin{aligned}&\operatorname{ord}_p |K_2^*(A/p[\pi])| \\ &\quad = f[(r_1 - 1)|\pi| - (r_1 - r_2)|\pi^p| - \cdots - (r_{e-1} - r_e)|\pi^{p^{e-1}}| - (r_e - 1)].\end{aligned}$$

Proof. *Step* 1. We first claim that

$$S_\pi(g) = \{h \in \pi : h^{p^m} \in \langle g \rangle\} \tag{1}$$

for any $m \geq 0$ and $g \in \pi^{p^m} - \pi^{p^{m+1}}$. Recall that $S_\pi(g)$ is the subgroup generated by all $h \in \pi$ such that $g \in \langle h \rangle$, then $\langle h^{p^m}, g \rangle$ is cyclic for such h, and $h^{p^m} \in \langle g \rangle$ since $g \notin \langle h^{p^{m+1}} \rangle$ by assumption. Conversely, if $h^{p^m} = g^a$ and $g = g_0^{p^m}$ for some $a \in \mathbb{Z}$ and $g_0 \in \pi$, then $(hg_0^{1-a})^{p^m} = g$, and so

$$h = g_0^{a-1}(hg_0^{1-a}) \in S_\pi(g).$$

Step 2. Again, consider the map

$$\Phi: \mathscr{U}(A\pi) \to \mathscr{U}(A\pi) \qquad (\Phi(h \otimes \lambda g) = h \otimes \varphi(\lambda) g^p).$$

Since π is abelian,

$$\mathscr{U}(A\pi) \cong \sum_{g\in\pi} (\pi/S_\pi(g)) \otimes A(g).$$

So ${}_p\mathscr{U}(A\pi)$ is generated by elements of the form $h \otimes \lambda g$, where $h^p \in S_\pi(g)$. Fix such an element, and assume $g \in \pi^{p^m} - \pi^{p^{m+1}}$ $(m \geqslant 0)$. Then $h^{p^{m+1}} = g^a$ for some a, by (1), and $p \mid a$ since $g \notin \pi^{p^{m+1}}$. So $h^{p^{m+1}} \in \langle g^p \rangle$, $g^p \in \pi^{p^n} - \pi^{p^{n+1}}$ for some $n \geqslant m+1$, and $h \in S_\pi(g^p)$ by (1) again. In other words, $\Phi(h \otimes \lambda g) = 0$, and we have shown that

$$_p\mathscr{U}(A\pi) \subseteq \mathrm{Ker}(\Phi). \tag{2}$$

We now claim that

$$\mathrm{Ker}(\Gamma_2^*\psi) = {}_p\mathscr{U}(A\pi). \tag{3}$$

Since $\Gamma_2^*\psi(x) = px - \Phi(x)$ by Proposition 6.1, $\mathrm{Ker}(\Gamma_2^*\psi)$ contains all p-torsion by (2). Conversely, consider any

$$x = \sum_{i=1}^{k} x_i \in \mathrm{Ker}(\Gamma_2^*\psi),$$

where each x_i is a sum of terms $h \otimes \lambda g$ with $|g| = p^i$. Then

$$\Gamma_2^*\psi(x) = px_k + \sum_{i=1}^{k-1} (px_i - \Phi(x_{i+1})) = 0,$$

so $px_k = 0$ and $px_i = \Phi(x_{i+1})$ for all $i < k$. By (2) and induction downwards from k, we get $px_i = 0$ for all i, and hence $x \in {}_p\mathscr{U}(A\pi)$.

Then, for all $n \geqslant 1$,

$$\mathrm{Ker}(p^{n-1}\Gamma_2^*\psi) = {}_{p^n}\mathscr{U}(A\pi).$$

Since $\eta_{A\pi} = 0$ by Theorem 4.2, Proposition 6.2 implies

$$\begin{aligned}
|K_2^*(A/p^n[\pi])| &= |H_2(\pi)| \cdot |\mathscr{U}_0(A\pi)/\mathrm{Im}(p^{n-1}\Gamma_2^*\psi)| \\
&= |\mathscr{U}(A\pi)/\mathrm{Im}(p^{n-1}\Gamma_2^*\psi)| = |\mathrm{Ker}(p^{n-1}\Gamma_2^*\psi)| \qquad (4)\\
&= |{}_{p^n}\mathscr{U}(A\pi)| = |\mathscr{U}(A\pi) \otimes \mathbb{Z}/p^n| = |\mathscr{U}(\pi) \otimes \mathbb{Z}/p^n|^f
\end{aligned}$$

(recall that $\mathscr{U}(A\pi) \cong A \otimes \mathscr{U}(\pi)$, and $f = [A : \hat{\mathbb{Z}}_p]$).

Step 3. It remains only to compute the order of

$$\mathscr{U}(\pi) \otimes \mathbb{Z}/p \cong \sum_{g\in\pi} (\pi/S_\pi(g)) \otimes \mathbb{Z}/p.$$

If $g \in \pi^{p^m} - \pi^{p^{m+1}}$ for any $0 \leqslant m < e$ ($p^e = \exp(\pi)$), and $g = g_0^{p^m}$, then

$$S_\pi(g) = \{h \in \pi : h^{p^m} \in \langle g \rangle\} = \langle g_0; x : x^{p^m} = 1 \rangle$$

by (1). In other words, $\pi/S_\pi(g)$ has rank one less than the number of cyclic summands of order at least $\mathbb{Z}/p^{m+1}$:

$$\operatorname{ord}_p |(\pi/S_\pi(g)) \otimes \mathbb{Z}/p| = r_{m+1} - 1,$$

where $r_{m+1} = p^{m+1} - rk(\pi)$. So

$$\begin{aligned}
&\operatorname{ord}_p |\mathscr{U}(\pi) \otimes \mathbb{Z}/p| \\
&\quad = \sum_{m=0}^{e-1} (r_{m+1} - 1)(|\pi^{p^m}| - |\pi^{p^{m-1}}|) \\
&\quad = r_1 \cdot |\pi| - \sum_{m=1}^{e-1} (r_m - r_{m+1}) |\pi^{p^m}| - r_e |\pi^{p^e}| - (|\pi| - 1) \\
&\quad = (r_1 - 1) |\pi| - \sum_{m=1}^{e-1} (r_m - r_{m+1}) |\pi^{p^m}| - (r_e - 1).
\end{aligned}$$

Combining this with (4) gives the formula for $|K_2^*(A/p[\pi])|$. ∎

The formula

$$|\mathscr{U}(A\pi)/\operatorname{Im}(\Gamma_2^* \psi)| = |\mathscr{U}(A\pi) \otimes \mathbb{Z}/p|$$

does, in fact, fail in general for non-abelian p-groups.

The techniques in [19] and [4] will now be used to find upper bounds for $|K_2(A/p[\pi])|$. The following symbol identities are in essence contained in Sections II.3 and III.4 of [19].

PROPOSITION 6.4. *Let p be a prime, and let R be a commutative ring of characteristic p. Fix $a \in R^*$.*

(i) *If $t \in R$ is such that $t^{mp^k} = 0$ for some $k \geqslant 1$ and $p \nmid m$, then*

$$\{1 + t, 1 + a^{p^k} t^{mp^k - 1}\} = 1 \qquad \text{in} \quad K_2(R).$$

(ii) *If $t \in R$ is such that $t^{mp^k + 1} = 0$ for some $k \geqslant 1$ and $p \nmid m$, and $l < k$, then*

$$\{a, 1 + a^{p^l} t^{mp^k}\} = 1 \qquad \textit{in} \quad K_2(R).$$

Proof. Set $A = \mathbb{F}_p[a, a^{-1}]$ and $A_\infty = A[[t]]$. Then $A[t]$ and A_∞ are regular rings by the Hilbert *syzygy* theorem [2, Sect. XII.2], and so by localization and devissage results in [23, Theorem 4 and 5], there are exact sequences

$$\begin{tikzcd} K_2(A) \arrow[r] \arrow[d, "="] & K_2(A[t]) \arrow[r, "i_1"] \arrow[d] & K_2(A[t, t^{-1}]) \arrow[d] \\ K_2(A) \arrow[r] & K_2(A_\infty) \arrow[r, "i_2"] & K_2(A_\infty[t^{-1}]) \end{tikzcd}$$

Since i_1 is injective by the corollary to Theorem 8 in [23], i_2 must also be injective.

Fix integers k, $m \geqslant 1$ and $0 \leqslant l \leqslant k$. Then relations of the form $\{u, 1-u\} = 1$ in $K_2(A_\infty[t^{-1}])$ give

$$\begin{aligned} &\{(-1)^{mp^{k-l}+1}a, 1 + a^{p^l}t^{mp^k}\} \\ &= \{(-1)^{mp^{k-l}+1}a, 1 + at^{mp^{k-l}}\}^{p^l} \\ &= \{(-t)^{mp^{k-l}}, 1 + at^{mp^{k-l}}\}^{-p^l} = \{(-t)^{mp^k}, 1 + at^{mp^{k-l}}\}^{-1} \\ &= \{-t, 1 + a^{p^k}t^{mp^{2k-l}}\}^{-m} = \{-t, (1+t)(1 + a^{p^k}t^{mp^{2k-l}})\}^{-m} \\ &= \{1 + (1+t)\,a^{p^k}t^{mp^{2k-l}-1}, (1+t)(1 + a^{p^k}t^{mp^{2k-l}})\}^m. \end{aligned}$$

So by injectivity of i_2,

$$\begin{aligned} &\{1 + (1+t)\,a^{p^k}t^{mp^{2k-l}-1}, (1+t)(1 + a^{p^k}t^{mp^{2k-l}})\}^m \\ &\quad = \{(-1)^{m+1}\,a, 1 + a^{p^l}t^{mp^k}\} \qquad \text{in} \quad K_2(A_\infty). \end{aligned} \tag{1}$$

Now consider the individual formulas to be shown.

(i) It suffices to prove this when $R = A_\infty/(t^{mp^k})$. When $l = k$, (1) reduces to the formula

$$\{1 + a^{p^k}t^{mp^k-1}, 1+t\}^m = 1 \qquad \text{in} \quad K_2(R).$$

Since $p \nmid m$ and $(1 + a^{p^k}t^{mp^k-1})^p = 1$ in R, this gives

$$\{1+t, 1 + a^{p^k}t^{mp^k-1}\} = 1.$$

(ii) It suffices to show this when $R = A_\infty/(t^{mp^k+1})$. Since $k > l \geqslant 0$, $t^{mp^{2k-l}-1} = 0$ in R. So by (1),

$$\{(-1)^{m+1}\,a, 1 + a^{p^l}t^{mp^k}\} = 1 \qquad \text{in} \quad K_2(R).$$

But $\{-1, 1 + a^{p^l}t^{mp^k}\} = 1$ since $(1 + a^{p^l}t^{mp^k})^p = 1$ in R (and $-1 = 1$ if $p = 2$). ∎

Now consider the groups

$$\Phi_n(R)=\mathrm{Ker}[K_2(R[t]/(t^{n+1}))\to K_2(R[t]/(t^n))],$$

defined for any ring R and $n\geqslant 1$.

LEMMA 6.5. *Let π be an abelian p-group, and set $r=rk(\pi)$. Then, for any $q=p^f$ and any $n\geqslant 1$,*

$$\begin{aligned}\mathrm{ord}_p\,|\Phi_n(\mathbb{F}_q\pi)| &\leqslant fr\,|\pi| && \text{if } n\not\equiv -1, 0\ (\mathrm{mod}\,p),\\ &\leqslant fr\,|\pi|+f(|\pi|-|\pi^{p^k}|) && \text{if } n=mp^k-1, k\geqslant 1, p\nmid m,\\ &\leqslant fr\,|\pi|-f(|\pi|-|\pi^{p^k}|) && \text{if } n=mp^k, k\geqslant 1, p\nmid m.\end{aligned}$$

Proof. Set $R=\mathbb{F}_q\pi[t]/(t^{n+1})$. Let $\{g_1,..., g_r\}$ be generators for π, and let

$$J=\langle g_1-1, g_2-1,..., g_r-1, t\rangle$$

denote the Jacobson radical of R. For convenience, set

$$K_2(R, I)=\mathrm{Ker}[K_2(R)\to K_2(R/I)]$$

for any ideal $I\subseteq R$. By Proposition 1.1 in [17],

$$\begin{aligned}\Phi_n(\mathbb{F}_q\pi)&=K_2(R, t^n)\\ &=\langle\{g_i, 1+yt^n\}, \{1+t, 1+yt^n\}\mid 1\leqslant i\leqslant r, y\in\mathbb{F}_q\pi\rangle. \qquad (1)\end{aligned}$$

We now consider the individual cases.

(i) Assume $n\not\equiv -1, 0\ (\mathrm{mod}\,p)$. For all $y\in\mathbb{F}_q\pi$,

$$\{1+t, 1+yt^n\}^{n+1}=1 \quad \text{in} \quad K_2(R),$$

by [20, Corollary 1.5b]. So $\{1+t, 1+yt^n\}=1$ since $(1+yt^n)^p=1$ in R, and $\Phi_n(\mathbb{F}_q\pi)$ is generated by all $\{g_i, 1+yt^n\}$. Hence,

$$\mathrm{ord}_p\,|\Phi_n(\mathbb{F}_q\pi)|\leqslant r\cdot rk_{\mathbb{F}_p}(\mathbb{F}_q\pi)=fr\,|\pi|.$$

(ii) If $n=mp^k-1$ where $k\geqslant 1$ and $p\nmid m$, then by Proposition 6.4,

$$\{1+t, 1+\lambda a^{p^k}t^n\}=1$$

for any $\lambda\in\mathbb{F}_q$ and $a\in\pi$. So by (1), $\Phi_n(\mathbb{F}_q)$ is generated by all

$$\{g_i, 1+yt^n\}\ (1\leqslant i\leqslant r, y\in\mathbb{F}_q\pi)$$

and

$$\{1+t,\, 1+\lambda a t^n\} \quad (\lambda \in \mathbb{F}_q,\, a \in \pi - \pi^{p^k}).$$

It follows that

$$\operatorname{ord}_p |\Phi_n(\mathbb{F}_q\pi)| \leqslant fr\, |\pi| + f(|\pi| - |\pi^{p^k}|).$$

(iii) Finally, assume $n = mp^k$ where $k \geqslant 1$ and $p \nmid m$. Then

$$\{1+t,\, 1+yt^n\} = 1 \qquad \text{in} \quad K_2(R),$$

for all $y \in \mathbb{F}_q\pi$, by the same argument as in (i). So

$$\Phi_n(\mathbb{F}_q\pi) = \langle \{g_i,\, 1+\lambda a t^n\}\colon 1 \leqslant i \leqslant r,\, \lambda \in \mathbb{F}_q,\, a \in \pi \rangle. \tag{2}$$

By Proposition 6.4(ii),

$$\{a,\, 1+\lambda a^{p^l} t^n\} = 1 \qquad \text{in} \quad K_2(R),$$

for any $a \in \pi$, $\lambda \in \mathbb{F}_q$, and $l < k$. This provides f relations among the elements of (2) for each $a \in \pi - \pi^{p^k}$, and so

$$\operatorname{ord}_p |\Phi_n(\mathbb{F}_q\pi)| \leqslant fr\, |\pi| - f(|\pi| - |\pi^{p^k}|). \quad \blacksquare$$

Finally, Lemma 6.5 can be applied to show:

THEOREM 6.6. *For any abelian p-group π and any $q = p^f$,*

$$K_2(\mathbb{F}_q\pi) = K_2^*(\mathbb{F}_q\pi).$$

In particular, if A is the unramified p-ring of degree f over $\hat{\mathbb{Z}}_p$, there is a short exact sequence

$$0 \to H_2(\pi) \to K_2(\mathbb{F}_q\pi) \to \mathscr{U}_0(A\pi)/\langle h \otimes p\lambda g - h \otimes \varphi(\lambda)\, g^p \rangle \to 0, \tag{1}$$

which is (at least non-canonically) split.

Proof. Fix π, and write $\pi = \bar{\pi} \times \mathbb{Z}/p^n$, where $\bar{\pi}$ is a product of cyclic groups of order at *least* p^n. Let $g \in \pi$ generate $\mathbb{Z}/p^n$, and set $t = g - 1$. Then

$$\mathbb{F}_q\pi \cong \mathbb{F}_q\bar{\pi}[g]/(g^{p^n} - 1) \cong \mathbb{F}_q\bar{\pi}[t]/(t^{p^n}).$$

Applying Lemma 6.5, we get (where $r = rk(\bar{\pi})$)

$$\begin{aligned}\operatorname{ord}_p |K_2(\mathbb{F}_q\pi)| &= \operatorname{ord}_p |K_2(\mathbb{F}_q\bar{\pi})| + \sum_{i=1}^{p^n-1} \operatorname{ord}_p |\Phi_i(\mathbb{F}_q\bar{\pi})\\ &\leqslant \operatorname{ord}_p |K_2(\mathbb{F}_q\bar{\pi})| + (p^n - 1)\, fr\, |\bar{\pi}| + f(|\bar{\pi}| - |\bar{\pi}^{p^n}|).\end{aligned}$$

In particular, $K_2(\mathbb{F}_q\pi) = 1$ if $\bar{\pi} = 1$.

If $\bar{\pi} \neq 1$, set

$$p^e = \exp(\bar{\pi}) = \exp(\pi), \qquad r_i = p^i\text{-}rk(\bar{\pi}).$$

Then $r_1 = r_2 = \cdots = r_n = r$, and $p^i\text{-}rk(\pi) = r+1$ for $i \leqslant n$. We may assume inductively that $K_2(\mathbb{F}_q\bar{\pi}) = K_2^*(\mathbb{F}_q\bar{\pi})$. So by Proposition 6.3,

$$\begin{aligned}
&\operatorname{ord}_p |K_2(\mathbb{F}_q\pi)| \\
&\quad \leqslant \operatorname{ord}_p |K_2(\mathbb{F}_q\bar{\pi})| + (p^n - 1)\, fr\, |\bar{\pi}| + f(|\bar{\pi}| - |\bar{\pi}^{p^n}|) \\
&\quad = f\,[(r-1)\,|\bar{\pi}| - (r_n - r_{n+1})\,|\bar{\pi}^{p^n}| - \cdots - (r_{e-1} - r_e)\,|\bar{\pi}^{p^{e-1}}| - (r_e - 1)] \\
&\qquad + fr(|\pi| - |\bar{\pi}|) + f\,|\bar{\pi}| - f\,|\bar{\pi}^{p^n}| \\
&\quad = f\,[r\,|\pi| - (r_n + 1 - r_{n+1})\,|\bar{\pi}^{p^n}| - \cdots - (r_{e-1} - r_e)\,|\bar{\pi}^{p^{e-1}}| - (r_e - 1)] \\
&\hspace{30em} \text{if } e > n, \\
&\quad = f\,[r\,|\pi| - r_e] \qquad \text{if } e = n.
\end{aligned}$$

In both cases, we get $|K_2(\mathbb{F}_q\pi)| \leqslant |K_2^*(\mathbb{F}_q\pi)|$ (applying Proposition 6.3 again), and the two groups are thus equal. The short exact sequence (1) now follows from Proposition 6.2 (and Theorem 4.2).

The construction of a splitting map $K_2(\mathbb{F}_q\pi) \to H_2(\pi)$ for (1) is easily reduced to the case where $\pi \cong \mathbb{Z}/p^n \times \mathbb{Z}/p^n$ ($n \geqslant 1$). By Proposition 6.1 and Theorems 3.6 and 4.2, $K_2^*(\mathbb{F}_q\pi)$ is a subquotient of $(A \otimes H_2(\pi)) \oplus (\pi \otimes A\pi)$, and hence in this case has exponent p^n. But any inclusion of $H_2(\pi) \cong \mathbb{Z}/p^n$ into a finite abelian group of exponent p^n splits. ∎

Theorem 6.6 generalizes automatically to give a description of $K_2(F\pi)$ for any finite field F and any finite abelian group π:

THEOREM 6.7. *Let F be any finite field, and let π be any finite abelian group. Set $p = \operatorname{char}(F)$, and let A be the unramified p-ring such that $F \cong A/p$ (i.e., $[A\colon \hat{\mathbb{Z}}_p] = [F\colon \mathbb{F}_p]$). Let φ be the Frobenius automorphism of A. Then there is a (non-canonical) isomorphism*

$$K_2(F\pi) \cong \frac{\pi \otimes A\pi}{\langle g \otimes \lambda g\colon g \in \pi, \lambda \in A \rangle \cdot \langle h \otimes (p\lambda g - \varphi(\lambda)\, g^p)\colon g, h \in \pi, \lambda \in A \rangle}.$$

Proof. Assume first that π is a p-group. Consider the short exact sequence

$$\begin{aligned}
0 &\to \frac{\mathcal{U}_0(A\pi)}{\langle h \otimes (p\lambda g - \varphi(\lambda)\, g^p)\rangle} \\
&\to \frac{\mathcal{U}(A\pi)}{\langle h \otimes (p\lambda g - \varphi(\lambda)\, g^p)\rangle} \xrightarrow{\ \nu\ } H_2(\pi) \to 0.
\end{aligned}$$

Write $\pi = \langle g_1 \rangle \times \cdots \times \langle g_k \rangle$, where $|g_1| \leqslant \cdots \leqslant |g_k|$. Fix $\lambda_0 \in A$ such that $\mathrm{Tr}(\lambda_0) = 1$. Then v is split by the map which sends $g_i \wedge g_j$ to $g_i \otimes \lambda_0 g_j$ for each $1 \leqslant i < j \leqslant k$. Together with Theorem 6.6, this shows that

$$\begin{aligned} K_2(F\pi) &\cong \frac{\mathscr{U}(A\pi)}{\langle h \otimes (p\lambda g - \varphi(\lambda)\, g^p)\rangle} \\ &\cong \frac{\pi \otimes A\pi}{\langle g \otimes \lambda g^n \rangle \cdot \langle h \otimes (p\lambda g - \varphi(\lambda)\, g^p)\rangle} \\ &= \frac{\pi \otimes A\pi}{\langle g \otimes \lambda g \rangle \cdot \langle h \otimes (p\lambda g - \varphi(\lambda)\, g^p)\rangle} \end{aligned} \tag{1}$$

in the p-group case. (To see the last step, note that

$$g \otimes \lambda g^{p^k} \equiv g \otimes p^k \varphi^{-k}(\lambda)\, g \qquad (\mathrm{mod} \langle h \otimes (p\lambda g - \varphi(\lambda)\, g^p)\rangle)$$

for any $k \geqslant 1$.)

Now let π be an arbitrary finite abelian group, and write $\pi = \sigma \times \bar{\pi}$, where $\bar{\pi}$ is a p-group and $p \nmid |\sigma|$. Let $K \supseteq A$ be the quotient field. Then $K\sigma \cong \sum_{i=1}^{s} K_i$ for some fields K_i which are unramified over $\hat{\mathbb{Q}}_p$. Let $A_i \subseteq K_i$ be the rings of integers and set $F_i = A_i/p$. Let

$$f\colon A\sigma \xrightarrow{\cong} \sum_{i=1}^{s} A_i, \qquad f'\colon A\pi \xrightarrow{\cong} \sum_{i=1}^{s} A_i\bar{\pi},$$

and

$$f''\colon \pi \otimes A\pi \cong \bar{\pi} \otimes A\pi \xrightarrow{\cong} \sum_{i=1}^{s} \bar{\pi} \otimes A_i\bar{\pi}$$

be the induced isomorphisms. Also, $K_2(F\pi) \cong \sum_{i=1}^{s} K_2(F_i\bar{\pi})$, and so by (1) we will be done upon showing that

$$f''(\langle g \otimes \lambda g\colon g \in \pi,\, \lambda \in A \rangle) = \sum_{i=1}^{s} \langle g \otimes \lambda g\colon g \in \bar{\pi},\, \lambda \in A_i \rangle \tag{2}$$

and ($\varphi_i \in \mathrm{Gal}(K_i/\hat{\mathbb{Q}}_p)$ denotes the Frobenius automorphism)

$$f'(\langle p\lambda g - \varphi(\lambda)\, g^p \rangle) = \sum_{i=1}^{s} \langle p\lambda g - \varphi_i(\lambda)\, g^p \rangle. \tag{3}$$

That (2) holds is clear. That (3) holds will follow once we check that for any $\lambda \in A$ and $g \in \sigma$,

$$(\Sigma\varphi_i) \circ f(\lambda g) = f(\varphi(\lambda)\, g^p). \tag{4}$$

But $f^{-1}(\Sigma\varphi_i)f$ and $(\lambda g \mapsto \varphi(\lambda)\, g^p)$ are both automorphisms of $A\sigma$, and both congruent mod p to the map $x \to x^p$. Two ring automorphisms of $A\sigma \cong A_i$ which are congruent mod p must agree on roots of unity of order prime to p, and hence must be equal. So (4) holds, and we are done. ∎

In fact, Theorem 6.6, together with the results and techniques in [11] and [22], can be used to give a description of $K_2(F\pi)$ when F is a finite field of characteristic p and π is any finite group with abelian p-Sylow subgroup.

Theorem 6.6 also give rise to the following obvious conjecture:

Conjecture 6.8. For any p-group π and any $q=p^f$,

$$K_2(\mathbb{F}_q\pi)=K_2^*(\mathbb{F}_q\pi).$$

7

Our original motivation for studying $K_2(\hat{\mathbb{Z}}_p\pi)$ was its appearance in localization sequences for computing $Cl_1(\mathbb{Z}\pi)$ and $SK_1(\mathbb{Z}\pi)$. Theorem 3.6, while not enough to give a complete algorithm for computing $Cl_1(\mathbb{Z}\pi)$, does motivate a conjecture for such an algorithm (for computing odd torsion in $Cl_1(\mathbb{Z}\pi)$, at least), as well as giving upper and lower bounds for $Cl_1(\mathbb{Z}\pi)$.

Fix an odd prime p and a p-group π. For each (complex) irreducible character χ of π, let $\hat{V}_\chi$ be the corresponding irreducible $\mathbb{C}\pi$-representation. By [24, Sects. 2 and 3], $\mathbb{Q}\chi$ has Schur index $m_\chi=1$, and $\mathbb{Q}\chi$ (the extension of $\mathbb{Q}$ by all $\chi(g)$) is isomorphic to the cyclotomic field $\mathbb{Q}\zeta_{p^n}$ for some n. Hence, $\hat{V}_\chi \cong \mathbb{C} \otimes_{\mathbb{Q}\chi} V_\chi$ for some irreducible $\mathbb{Q}\chi[\pi]$-module V_χ, and the simple summand A_χ of $\mathbb{Q}\pi$ corresponding to χ has the form

$$A_\chi \cong \operatorname{End}_{\mathbb{Q}\chi}(V_\chi) \cong M_{r_\chi}(\mathbb{Q}\chi) \qquad (r_\chi=\dim_{\mathbb{Q}\chi}(V_\chi)).$$

For shortness of notation, we let $\mathbb{Z}\chi$ and $\hat{\mathbb{Z}}_p[\chi]$ denote the rings of integers in $\mathbb{Q}\chi$ and $\hat{\mathbb{Q}}_p[\chi]$, respectively.

For each χ, let $(\mu_{\mathbb{Q}\chi})_p$ denote the group of p-power roots of unity in $\mathbb{Q}\chi$. For any set T of irreducible characters of π, homomorphisms

$$\varphi_T=\prod_{\chi\in T}\varphi_\chi\colon K_2^{\mathrm{top}}(\hat{\mathbb{Z}}_p\pi)\to\prod_{\chi\in T}K_2^{\mathrm{top}}(\hat{\mathbb{Z}}_p[\chi]),$$

$$\psi_T=\prod_{\chi\in T}\psi_\chi\colon H_1(\pi;\hat{\mathbb{Z}}_p\pi)\to\prod_{\chi\in T}(\mu_{\mathbb{Q}\chi})_p,$$

and

$$\lambda_T=\prod_{\chi\in T}\lambda_\chi\colon\prod_{\chi\in T}K_2^{\mathrm{top}}(\hat{\mathbb{Z}}_p[\chi])\xrightarrow{\cong}\prod_{\chi\in T}(\mu_{\mathbb{Q}\chi})_p$$

are defined as follows. Fix $\chi \in T$, and let $M_\chi \subseteq V_\chi$ be a π-invariant $\mathbb{Z}\chi$-lattice. Let φ_χ be the composite

$$\begin{aligned}\varphi_\chi\colon K_2^{\mathrm{top}}(\hat{\mathbb{Z}}_p\pi) &\to K_2^{\mathrm{top}}(\hat{\mathbb{Z}}_p \otimes \mathrm{End}_{\mathbb{Z}\chi}(M_\chi))\\ &\cong K_2^{\mathrm{top}}(M_{r_\chi}(\hat{\mathbb{Z}}_p[\chi])) \cong K^{\mathrm{top}}(\hat{\mathbb{Z}}_p[\chi]).\end{aligned}$$

For any commuting elements $g, h \in \pi$, set

$$\psi_\chi(g \otimes h) = \det_{\mathbb{Q}\chi}(g, V_\chi^h) \in (\mu_{\mathbb{Q}\chi})_p,$$

where $V_\chi^h \subseteq V_\chi$ is the subspace of elements fixed by h. Finally, λ_χ denotes the norm residue symbol map $\lambda_\chi(\{a, b\}) = (a, b)_v$ (and is an isomorphism by [20, Theorem 5.1]).

PROPOSITION 7.1. *For any irreducible character χ of π, and any commuting g, $h \in \pi$,*

$$\psi_\chi(g \otimes h) = \lambda_\chi \circ \varphi_\chi(\{g, \Gamma^{-1}(\tfrac{1}{2}(h + h^{-1}))\}). \tag{1}$$

For any $g \in \pi$ and $u \in (\hat{\mathbb{Z}}_p[Z_\pi(g)])^*$,

$$\psi_\chi(g \otimes \Gamma(u)) = \lambda_\chi \circ \varphi_\chi(\{g, u\}). \tag{2}$$

Proof. Set $q = p^k = \exp(\pi)$. Let

$$A_\chi \cong \mathrm{End}_{\mathbb{Q}\chi}(V_\chi) \cong M_r(\mathbb{Q}\chi) \subseteq M_r(\mathbb{Q}\zeta_q) \qquad (r = r_\chi = \dim_{\mathbb{Q}\chi}(V_\chi))$$

be the simple summand of $\mathbb{Q}\pi$ corresponding to χ. If $g, h \in \pi$ and $[g, h] = 1$, then let $\alpha(g)$ and $\alpha(h)$ denote their images in $M_r(\mathbb{Q}\chi)$. Since $\langle g, h\rangle$ is an abelian group of exponent dividing q, the matrices $\alpha(g)$ and $\alpha(h)$ conjugate (simultaneously) in $M_r(\mathbb{Q}\zeta_q)$ to diagonal matrices

$$\begin{aligned}\alpha'(g) &= \mathrm{diag}(\mu_1, \ldots, \mu_r),\\ \alpha'(h) &= \mathrm{diag}(\nu_1, \ldots, \nu_r) \qquad (\mu_i, \nu_i \in \langle\zeta_q\rangle).\end{aligned}$$

Write

$$\Gamma^{-1}(\tfrac{1}{2}(h + h^{-1})) = \sum_j \lambda_j h^j \in (\hat{\mathbb{Z}}_p[\langle h\rangle])^*.$$

Then, since $K_2(\hat{\mathbb{Z}}_p[\chi])$ injects into $K_2(\hat{\mathbb{Z}}_p\zeta_q)$ (see [20]),

$$\begin{aligned}\varphi_\chi(\{g, \Gamma^{-1}(\tfrac{1}{2}(h + h^{-1}))\}) &= \Big\{\alpha(g), \sum_j \lambda_j \alpha(h)^j\Big\}\\ &= \prod_{i=1}^{r} \Big\{\mu_i, \sum_j \lambda_j \nu_i^j\Big\} \in K_2(\hat{\mathbb{Z}}_p\zeta_q).\end{aligned}$$

Also, taking norm residue symbols commutes with inclusions of cyclotomic fields $\hat{\mathbb{Q}}_p\zeta_{p^i}$ for odd p (this follows, e.g., from the formulas in [18]). So by [17, Lemma 1.4],

$$\lambda_\chi \circ \varphi_\chi(\{g, \Gamma^{-1}(\tfrac{1}{2}(h+h^{-1}))\}) = \prod_{i=1}^{r} \left(\mu_i, \sum_j \lambda_j v_i^j\right)_v = \prod_{i=1}^{r} \mu_i^{\tau(v_i)},$$

where $\tau(v_i)=1$ if $v_i=1$, $\tau(v_i)=0$ if $v_i \neq 1$. But the last expression is just $\det(g, V_\chi^h)$, and so (1) is proved.

To show (2), fix $g \in \pi$ and $u \in (\hat{\mathbb{Z}}_p[\hat{Z}_\pi(g)])^*$, and let $\bar{u}$ be the unit obtained from u by replacing $h \mapsto h^{-1}$. Then

$$\lambda_\chi \circ \varphi_\chi(\{g, u\}) = \lambda_\chi \circ \varphi_\chi(\{g, \bar{u}\}):$$

$(\zeta_{p^k}, a)_v = (\zeta_{p^k}, \bar{a})_v$ for any $a \in (\hat{\mathbb{Z}}_p\zeta_{p^k})^*$ by [18]. So by (1),

$$\begin{aligned}\lambda_\chi \circ \varphi_\chi(\{g, u\}) &= \lambda_\chi \circ \varphi_\chi(\{g, u\bar{u}\})^{1/2} = \psi_\chi(g \otimes \Gamma(u\bar{u}))^{1/2} \\ &= \psi_\chi(g \otimes \tfrac{1}{2}(\Gamma(u) + \Gamma(\bar{u}))) = \psi_\chi(g \otimes \Gamma(u)):\end{aligned}$$

the last step since $\psi_\chi(g \otimes h) = \psi_\chi(g \otimes h^{-1})$ $(V_\chi^h = V_\chi^{h^{-1}})$. ∎

We use the term a cluster of characters for a p-group π to mean any set of irreducible characters which contains exactly one character dividing each irreducible $\mathbb{Q}\pi$-module. Thus in the above notation, if S is a cluster of characters for π, then there is a natural isomorphism

$$\mathbb{Q}\pi \cong \prod_{\chi \in S} M_{r_\chi}(\mathbb{Q}\chi).$$

Furthermore, for each $\chi \in S$,

$$K_2^{\mathrm{top}}(\hat{\mathbb{Z}}_p[\chi]) \cong (\mu_{\mathbb{Q}\chi})_p \cong \varprojlim_n SK_1(\mathbb{Z}\chi, p^n)$$

by [20, Sect. 5]. So Proposition 1.8 in [11] takes the form

$$Cl_1(\mathbb{Z}\pi) \cong \mathrm{Coker}(\varphi_S).$$

Since $\mathrm{Im}(\psi_S) \subseteq \mathrm{Im}(\lambda_S \circ \varphi_S)$ by Proposition 7.1, this implies:

THEOREM 7.2. *Let p be an odd prime, let π be a p-group, and let S be a cluster of characters for π. Then λ_S^{-1} induces a surjection*

$$\Lambda_S\colon \mathrm{Coker}(\psi_S) \twoheadrightarrow Cl_1(\mathbb{Z}\pi).$$

The following conjecture is motivated partly since it holds in the abelian case [17, Theorem 1.8], partly by the similarity between $H_1(\pi; \hat{\mathbb{Z}}_p\pi)$ and

$K_2^*(\hat{\mathbb{Z}}_p\pi)$ (Theorem 3.6), and partly since it seems to be the best hope for finding a "reasonable" combinatorial algorithm for computing $Cl_1(\mathbb{Z}\pi)$.

Conjecture 7.3. For any odd prime p, any p-group π, and any cluster S of characters for π,

$$\Lambda_S\colon \mathrm{Coker}(\psi_S) \to Cl_1(\mathbb{Z}\pi)$$

is an isomorphism.

Note that if Conjecture 7.3 holds, then this together with [11, Theorem 4.8(ii)] and [22, Theorem 2(i)] gives a direct combinatorial algorithm for computing odd torsion in $SK_1(\mathbb{Z}\pi)$ for an arbitrary finite group π. Hopefully, if Conjecture 7.3 could be shown, then a similar but more complicated algorithm could be found for computing $Cl_1(\mathbb{Z}\pi)$ when π is a 2-group (see Theorem 1.8 in [17] for the abelian case).

The best we can do so far towards proving Conjecture 7.3 is to show that $\mathrm{Im}(\lambda_{S'} \circ \varphi_{S'}) = \mathrm{Im}(\psi_{S'})$ when S' is a set of *linear* characters for π: for example, a cluster of characters for π^{ab}. This does at least give a lower bound for $Cl_1(\mathbb{Z}\pi)$, i.e., it is at least as large as $\mathrm{Coker}(\psi_{S'})$ when S' is such a cluster of characters.

In fact, this result can be extended to include 2-groups. If π is a 2-group and T any set of linear characters for π, then

$$\psi_T = \prod \psi_\chi\colon H_1(\pi; \hat{\mathbb{Z}}_2\pi) \to \prod_{\chi \in T} (\mu_{\mathbb{Q}\chi})_2$$

is defined by setting, for $\chi \in T$ and commuting $g, h \in \pi$,

$$\begin{aligned} \psi_\chi(g \otimes h) &= \chi(g) \qquad \text{if} \quad \chi(h) = 1, \\ &= 1 \qquad\quad\ \ \text{if} \quad \chi(h) \neq 1. \end{aligned}$$

Note that this coincides with the definition of ψ_T for odd p-groups—in the case of linear characters.

When T is a set of linear characters for π such that for all $\chi \in T$, $\mathbb{Q}\chi = \mathbb{Q}\zeta_{2^n}$ for some $n \geqslant 2$, then a modified norm residue symbol map

$$\lambda'_T = \prod \lambda'_\chi\colon \prod_{\chi \in T} K_2^{\mathrm{top}}(\hat{\mathbb{Z}}_2[\chi]) \xrightarrow{\cong} \prod_{\chi \in T} (\mu_{\mathbb{Q}\chi})_2 = \prod_{\chi \in T} \mathrm{Im}(\chi)$$

is defined as follows: if $|\mathrm{Im}(\chi)| = 2^n$ and $a, b \in (\hat{\mathbb{Z}}_2[\chi])^*$, then

$$\lambda'_\chi(\{a, b\}) = (a, b)_v^{2^{n-1}+1}.$$

To simplify the statement of the next theorem, we set $\lambda'_T = \lambda_T$ whenever T is a set of characters for an odd p-group.

Finally, for any 2-group π, let

$$v\colon H_1(\pi; \hat{\mathbb{Z}}_2\pi) \to H_2(\pi)$$

be the map defined in Section 3: $v(g \otimes h) = g \wedge h \in H_2(\pi)$ for any pair of commuting elements $g, h \in \pi$.

THEOREM 7.4. *Let p be any prime, let π be a p-group, and let T be a cluster of characters for π^{ab}. Set $T_0 = T$ if p is odd;*

$$T_0 = \{\chi \in T\colon |\mathrm{Im}(\chi)| > 2\} = \{\chi \in T\colon \mathbb{Q}\chi \not\subseteq \mathbb{R}\}$$

if $p = 2$. Consider the maps

$$K_2^{\mathrm{top}}(\hat{\mathbb{Z}}_p\pi) \xrightarrow{\lambda'_{T_0} \circ \varphi_{T_0}} \prod_{\chi \in T_0} (\mu_{\mathbb{Q}\chi})_p = \prod_{\chi \in T_0} \mathrm{Im}(\chi)$$

$$\xleftarrow{\psi_{T_0}} H_1(\pi; \hat{\mathbb{Z}}_p\pi).$$

(i) *If p is odd, then $\mathrm{Im}(\lambda'_T \circ \varphi_T) = \mathrm{Im}(\psi_T)$, and there is a surjection*

$$Cl_1(\mathbb{Z}\pi) \twoheadrightarrow \left[\prod_{\chi \in T} \mathrm{Im}(\chi)\right] \Big/ \psi_T(H_1(\pi; \hat{\mathbb{Z}}_p\pi)).$$

(ii) *If $p = 2$, then*

$$\mathrm{Im}(\lambda'_{T_0} \circ \varphi_{T_0}) = \psi_{T_0}(\mathrm{Ker}[v\colon H_1(\pi; \hat{\mathbb{Z}}_2\pi) \to H_2(\pi)]),$$

and there is a surjection

$$Cl_1(\mathbb{Z}\pi) \twoheadrightarrow \left[\prod_{\chi \in T_0} \mathrm{Im}(\chi)\right] \Big/ \psi_{T_0}(\mathrm{Ker}[v\colon H_1(\pi; \hat{\mathbb{Z}}_2\pi) \to H_2(\pi)]).$$

Proof. Set

$$X = \langle \psi_{T_0}(g \otimes g^m)\colon g \in \pi, m \in \mathbb{Z} \rangle \subseteq \prod_{\chi \in T_0} \mathrm{Im}(\chi).$$

We first claim that

$$X \subseteq \psi_{T_0}(\mathrm{Ker}(v)) \cap \mathrm{Im}(\lambda'_{T_0} \circ \varphi_{T_0}) \tag{1}$$

and that

$$\langle \lambda'_{T_0} \circ \varphi_{T_0}(\{u, v\})\colon u, v \in (\hat{\mathbb{Z}}_p\sigma)^*, \text{ some cyclic } \sigma \subseteq \pi \rangle \subseteq X. \tag{2}$$

By the naturality of ψ and $\lambda' \circ \varphi$ (see the remarks before Proposition 1.5 in

[17]), it suffices to prove (1) and (2) for cyclic π. But in this case, $H_2(\pi)=0$; ψ_{T_0} and φ_{T_0} are onto since $SK_1(\mathbb{Z}\pi)=0$ (use Theorem 1.8 in [17]); $X=\mathrm{Im}(\psi_{T_0})$ by definition; and the result follows.

Now consider the following diagram:

$$\begin{tikzcd} & K_2^{\mathrm{top}}(\hat{\mathbb{Z}}_p\pi) \arrow[r, "K_2(\alpha)"] \arrow[d, "\Gamma_2^*"] & K_2^{\mathrm{top}}(\hat{\mathbb{Z}}_p[\pi^{ab}]) \arrow[d, "\Gamma_2^*"] \arrow[dr, "\bar{\varphi}"] & \\ \mathrm{Ker}(\nu) \arrow[r, two heads] & \mathscr{U}_0(\pi) \arrow[r, "\mathscr{U}_0(\alpha)"] & \mathscr{U}_0(\pi^{ab}) \arrow[r, "\bar{\psi}"'] & \Big[\prod_{\chi\in T_0}\mathrm{Im}(\chi)\Big]\Big/X \end{tikzcd} \tag{3}$$

Here, $\bar{\varphi}$ and $\bar{\psi}$ are the reductions of the maps $\lambda'\circ\varphi$ and ψ for π^{ab}: recall that

$$\mathscr{U}(\pi^{ab})=H_1(\pi^{ab};\hat{\mathbb{Z}}_p[\pi^{ab}])/\langle g\otimes g^m\colon g\in\pi^{ab},\ m\in\mathbb{Z}\rangle.$$

Also, $\alpha\colon\pi\to\pi^{ab}$ denotes the projection, and $K_2(\alpha)$ and $\mathscr{U}_0(\alpha)$ are the induced maps. The square in (3) commutes by naturality of Γ_2^*.

By Corollary 1.2 in [17], $K_2^{\mathrm{top}}(\hat{\mathbb{Z}}_p[\pi^{ab}])$ is generated by symbols $\{1+p, 1+px\}$ $(x\in\hat{\mathbb{Z}}_p[\pi^{ab}])$, $\{g,\lambda\}$ $(g\in\pi^{ab},\ \lambda\in(\hat{\mathbb{Z}}_p)^*)$, and $\{g,u\}$ $(g\in\pi^{ab},\ u\in 1+I(\hat{\mathbb{Z}}_p[\pi^{ab}]))$. That

$$\{1+p, 1+px\},\ \{g,\lambda\}\in\mathrm{Ker}(\bar{\varphi})\cap\mathrm{Ker}(\Gamma_2^*)$$

follows from (2) and the fact that Γ_2^* factors through $K_2^*(\hat{\mathbb{Z}}_p[\pi^{ab}])$ ($K_1(\hat{\mathbb{Z}}_p\pi,p)$ is generated by cyclic induction by, e.g., [10, Proposition 2]). On the other hand, for any g and u,

$$\Gamma_2^*(\{g,u\})=g\otimes\Gamma(u)$$

by Proposition 3.8, and by Lemma 1.4 in [17],

$$\bar{\varphi}(\{g,u\})=\bar{\psi}(g\otimes\Gamma(u)).$$

So the triangle in diagram (3) commutes.

Hence,

$$\begin{aligned}\mathrm{Im}(\lambda'_{T_0}\circ\varphi_{T_0})/X=\mathrm{Im}(\bar{\varphi}\circ K_2(\alpha))&=\mathrm{Im}(\bar{\psi}\circ\mathscr{U}_0(\alpha))\\ &=\psi_{T_0}(\mathrm{Ker}(\nu))/X.\end{aligned}$$

Together with (1), this shows that

$$\mathrm{Im}(\lambda'_{T_0}\circ\varphi_{T_0})=\psi_{T_0}(\mathrm{Ker}(\nu)). \tag{4}$$

So we are done if $p=2$. If p is odd, then $\mathrm{Im}(\lambda_T\circ\varphi_T)\supseteq\mathrm{Im}(\psi_T)$ by Proposition 7.1, and so they are equal by (4). ∎

Theorem 7.4 provides a means of detecting elements in $Cl_1(\mathbb{Z}\pi)$ which vanish in $Cl_1(\mathbb{Z}[\pi^{ab}])$. These are precisely the elements which were detected in Propositions 16 and 18 in [21], and in Step 7 of the proof of Theorem 5.6 in [11]. In fact, Theorem 7.4 and its proof give some explanation of why the proofs in Section 2 of [21] follow such a regular pattern.

We end with the following example of the use of Theorems 7.2 and 7.4:

THEOREM 7.5. *For any odd prime p and any non-abelian group π of order p^3,*

$$SK_1(\mathbb{Z}\pi) = Cl_1(\mathbb{Z}\pi) \cong (\mathbb{Z}/p)^{p-1}.$$

Proof. That $SK_1(\hat{\mathbb{Z}}_p\pi) = 0$ (and hence $SK_1(\mathbb{Z}\pi) = Cl_1(\mathbb{Z}\pi)$) follows from Proposition 9 in [10].

Note that $\pi^{ab} \cong \mathbb{Z}/p \times \mathbb{Z}/p$. Fix $a, b \in \pi$ which generate π^{ab}, then $z = [a, b]$ generates $[\pi, \pi]$. Furthermore,

$$\mathbb{Q}\pi \cong \mathbb{Q}[\pi^{ab}] \times M_p(\mathbb{Q}\zeta_p) \cong \mathbb{Q} \times (\mathbb{Q}\zeta_p)^{p+1} \times M_p(\mathbb{Q}\zeta_p).$$

So there is a cluster S of characters for π of the form

$$S = \{1, \chi_0, ..., \chi_p, \bar{\chi}\},$$

where $\bar{\chi}$ is a non-linear character and $\mathrm{Im}(\chi_i) = \langle \zeta_p \rangle$ for $0 \leqslant i \leqslant p$. Set $T = S - \{\bar{\chi}\}$, a cluster of characters for π^{ab}.

By Theorems 7.2 and 7.4, there are surjections

$$\mathrm{Coker}(\psi_S) \twoheadrightarrow Cl_1(\mathbb{Z}\pi) \twoheadrightarrow \mathrm{Coker}(\psi_T).$$

On the other hand, $\psi_\chi(z \otimes a) = 1$ for any $\chi \in T$ (since $\chi(z) = 1$), z acts on $V_{\bar{\chi}}$ via multiplication by ζ for some primitive pth root of unity ζ, and so $\psi_{\bar{\chi}}(z \otimes a) = \zeta$ since $V_{\bar{\chi}}^a$ is one-dimensional. It follows that

$$\mathrm{Coker}(\psi_S) \cong \mathrm{Coker}(\psi_T) \cong Cl_1(\mathbb{Z}\pi).$$

Finally, ψ_T factors as a composite

$$\psi_T\colon H_1(\pi; \hat{\mathbb{Z}}_p\pi) \to H_1(\pi^{ab}; \hat{\mathbb{Z}}_p[\pi^{ab}]) \to \prod_{\chi \in T} \mathrm{Im}(\chi) \cong (\mathbb{Z}/p)^{p+1}.$$

Since $H_1(\pi; \hat{\mathbb{Z}}_p\pi)$ is generated by elements $g \otimes h$ for commuting $g, h \in \pi$, it follows that $\mathrm{Im}(\psi_T)$ is generated by $\psi_T(a \otimes e)$ and $\psi_T(b \otimes e)$. These are clearly independent, and so

$$Cl_1(\mathbb{Z}\pi) \cong \mathrm{Coker}(\psi_T) \cong (\mathbb{Z}/p)^{p-1}. \quad \blacksquare$$

Note added in proof. (1) Conjecture 7.3—the proposed algorithm for describing $\mathrm{Cl}_1(\mathbb{Z}\pi)$ when π is a p-group and p is odd—has recently been proven.

(2) When p is any prime, π is an *abelian* p-group, and A is an unramified p-ring, a short exact sequence

$$0 \to K_2^{\mathrm{top}}(A\pi)/\langle\{\pm g, \pm h\}\rangle \xrightarrow{\Gamma_2} \frac{\pi \otimes A\pi}{\langle g \otimes \lambda g\rangle} \xrightarrow{\hat{\nu}} \frac{\pi \otimes \pi}{\langle g \otimes h + h \otimes g\rangle} \to 0$$

has now been constructed. Here, Γ_2 and $\hat{\nu}$ are liftings of the homomorphisms Γ_2^* and ν constructed in Section 3.

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